1

DEFINITION OF PURE MATHEMATICS

1. PURE Mathematics is the class of all propositions of the form "*p* implies *q*", where *p* and *q* are propositions containing one or more variables, the same in the two propositions, and neither *p* nor *q* contains any constants except logical constants. And logical constants are all notions definable in terms of the following: implication, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics *uses* a notion which is not a constituent of the propositions which it considers, namely the notion of truth.

2. The above definition of pure mathematics is, no doubt, somewhat unusual. Its various parts, nevertheless, appear to be capable of exact justification—a justification which it will be the object of the present work to provide. It will be shown that whatever has, in the past, been regarded as pure mathematics, is included in our definition, and that whatever else is included possesses those marks by which mathematics is commonly though vaguely distinguished from other studies. The definition professes to be, not an arbitrary decision to use a common word in an uncommon signification, but rather a precise analysis of the ideas which, more or less unconsciously, are implied in the ordinary employment of the term. Our method will therefore be one of analysis, and our problem may be called philosophical—in the sense, that is to say, that we seek to pass from the complex to the simple, from the demonstrable to its indemonstrable premisses. But in one respect not a few of our discussions will differ from those that are usually called philosophical. We shall be able, thanks to the labours of the mathematicians themselves, to arrive at certainty in regard to most of the questions with

which we shall be concerned; and among those capable of an exact solution we shall find many of the problems which, in the past, have been involved in all the traditional uncertainty of philosophical strife. The nature of number, of infinity, of space, time and motion, and of mathematical inference itself, are all questions to which, in the present work, an answer professing itself demonstrable with mathematical certainty will be given—an answer which, however, consists in reducing the above problems to problems in pure logic, which last will not be found satisfactorily solved in what follows.

3. The Philosophy of Mathematics has been hitherto as controversial, obscure and unprogressive as the other branches of philosophy. Although it was generally agreed that mathematics is in some sense true, philosophers disputed as to what mathematical propositions really meant: although something was true, no two people were agreed as to what it was that was true, and if something was known, no one knew what it was that was known. So long, however, as this was doubtful, it could hardly be said that any certain and exact knowledge was to be obtained in mathematics. We find, accordingly, that idealists have tended more and more to regard all mathematics as dealing with mere appearance, while empiricists have held everything mathematical to be approximation to some exact truth about which they had nothing to tell us. This state of things, it must be confessed, was thoroughly unsatisfactory. Philosophy asks of Mathematics: What does it mean? Mathematics in the past was unable to answer, and Philosophy answered by introducing the totally irrelevant notion of mind. But now Mathematics is able to answer, so far at least as to reduce the whole of its propositions to certain fundamental notions of logic. At this point, the discussion must be resumed by Philosophy. I shall endeavour to indicate what are the fundamental notions involved, to prove at length that no others occur in mathematics and to point out briefly the philosophical difficulties involved in the analysis of these notions. A complete treatment of these difficulties would involve a treatise on Logic, which will not be found in the following pages.

4. There was, until very lately, a special difficulty in the principles of mathematics. It seemed plain that mathematics consists of deductions, and yet the orthodox accounts of deduction were largely or wholly inapplicable to existing mathematics. Not only the Aristotelian syllogistic theory, but also the modern doctrines of Symbolic Logic, were either theoretically inadequate to mathematical reasoning, or at any rate required such artificial forms of statement that they could not be practically applied. In this fact lay the strength of the Kantian view, which asserted that mathematical reasoning is not strictly formal, but always uses intuitions, *i.e.* the *à priori* knowledge of space and time. Thanks to the progress of Symbolic Logic, especially as treated by Professor Peano, this part of the Kantian philosophy is now capable

of a final and irrevocable refutation. By the help of ten principles of deduction and ten other premisses of a general logical nature (*e.g.* "implication is a relation"), all mathematics can be strictly and formally deduced; and all the entities that occur in mathematics can be defined in terms of those that occur in the above twenty premisses. In this statement, Mathematics includes not only Arithmetic and Analysis, but also Geometry, Euclidean and non-Euclidean, rational Dynamics and an indefinite number of other studies still unborn or in their infancy. The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age; and when this fact has been established, the remainder of the principles of mathematics consists in the analysis of Symbolic Logic itself.

5. The general doctrine that all mathematics is deduction by logical principles from logical principles was strongly advocated by Leibniz, who urged constantly that axioms ought to be proved and that all except a few fundamental notions ought to be defined. But owing partly to a faulty logic, partly to belief in the logical necessity of Euclidean Geometry, he was led into hopeless errors in the endeavour to carry out in detail a view which, in its general outline, is now known to be correct.* The actual propositions of Euclid, for example, do not follow from the principles of logic alone; and the perception of this fact led Kant to his innovations in the theory of knowledge. But since the growth of non-Euclidean Geometry, it has appeared that pure mathematics has no concern with the question whether the axioms and propositions of Euclid hold of actual space or not: this is a question for applied mathematics, to be decided, so far as any decision is possible, by experiment and observation. What pure mathematics asserts is merely that the Euclidean propositions follow from the Euclidean axioms—*i.e.* it asserts an implication: any space which has such and such properties has also such and such other properties. Thus, as dealt with in pure mathematics, the Euclidean and non-Euclidean Geometries are equally true: in each nothing is affirmed except implications. All propositions as to what actually exists, like the space we live in, belong to experimental or empirical science, not to mathematics; when they belong to applied mathematics, they arise from giving to one or more of the variables in a proposition of pure mathematics some constant value satisfying the hypothesis, and thus enabling us, for that value of the variable, actually to assert both hypothesis and consequent instead of asserting merely the implication. We assert always in mathematics that if a certain assertion p is true of any entity x, or of any set of entities x, y, z, . . ., then some other assertion q is true of those entities; but we do not assert either p or q separately of our entities. We assert a relation between the assertions p and q, which I shall call *formal implication*.

* On this subject, cf. Couturat, *La Logique de Leibniz*, Paris, 1901.

6. Mathematical propositions are not only characterized by the fact that they assert implications, but also by the fact that they contain *variables*. The notion of the variable is one of the most difficult with which Logic has to deal, and in the present work a satisfactory theory as to its nature, in spite of much discussion, will hardly be found. For the present, I only wish to make it plain that there are variables in all mathematical propositions, even where at first sight they might seem to be absent. Elementary Arithmetic might be thought to form an exception: $1 + 1 = 2$ appears neither to contain variables nor to assert an implication. But as a matter of fact, as will be shown in Part II, the true meaning of this proposition is: "If x is one and y is one, and x differs from y, then x and y are two." And this proposition both contains variables and asserts an implication. We shall find always, in all mathematical propositions, that the words *any* or *some* occur; and these words are the marks of a variable and a formal implication. Thus the above proposition may be expressed in the form: "Any unit and any other unit are two units." The typical proposition of mathematics is of the form "ϕ $(x, y, z, \ldots)$ implies ψ $(x, y, z, \ldots)$, whatever values $x, y, z, \ldots$ may have"; where ϕ $(x, y, z, \ldots)$ and ψ $(x, y, z, \ldots)$, for every set of values of $x, y, z, \ldots$, are propositions. It is not asserted that ϕ is always true, nor yet that ψ is always true, but merely that, in all cases, when ϕ is false as much as when ϕ is true, ψ follows from it.

The distinction between a variable and a constant is somewhat obscured by mathematical usage. It is customary, for example, to speak of parameters as in some sense constants, but this is a usage which we shall have to reject. A constant is to be something absolutely definite, concerning which there is no ambiguity whatever. Thus 1, 2, 3, e, π, Socrates, are constants; and so are *man*, and the human race, past, present and future, considered collectively. Proposition, implication, class, etc. are constants; but a proposition, any proposition, some proposition, are not constants, for these phrases do not denote one definite object. And thus what are called parameters are simply variables. Take, for example, the equation $ax + by + c = 0$, considered as the equation to a straight line in a plane. Here we say that x and y are variables, while a, b, c are constants. But unless we are dealing with one absolutely particular line, say the line from a particular point in London to a particular point in Cambridge, our a, b, c are not definite numbers, but stand for *any* numbers, and are thus also variables. And in Geometry nobody does deal with actual particular lines; we always discuss *any* line. The point is that we collect the various couples x, y into classes of classes, each class being defined as those couples that have a certain fixed relation to one triad (a, b, c). But from class to class, a, b, c also vary, and are therefore properly variables.

7. It is customary in mathematics to regard our variables as restricted to certain classes: in Arithmetic, for instance, they are supposed to stand for numbers. But this only means that if they stand for numbers, they satisfy some

formula, *i.e.* the hypothesis that they are numbers implies the formula. This, then, is what is really asserted, and in this proposition it is no longer necessary that our variables should be numbers: the implication holds equally when they are not so. Thus, for example, the proposition "x and y are numbers implies $(x + y)^2 = x^2 + 2xy + y^2$" holds equally if for x and y we substitute Socrates and Plato:* both hypothesis and consequent, in this case, will be false, but the implication will still be true. Thus in every proposition of pure mathematics, when fully stated, the variables have an absolutely unrestricted field: any conceivable entity may be substituted for any one of our variables without impairing the truth of our proposition.

8. We can now understand why the constants in mathematics are to be restricted to logical constants in the sense defined above. The process of transforming constants in a proposition into variables leads to what is called generalization, and gives us, as it were, the formal essence of a proposition. Mathematics is interested exclusively in *types* of propositions; if a proposition p containing only constants be proposed, and for a certain one of its terms we imagine others to be successively substituted, the result will in general be sometimes true and sometimes false. Thus, for example, we have "Socrates is a man"; here we turn Socrates into a variable, and consider "x is a man". Some hypotheses as to x, for example, "x is a Greek", insure the truth of "x is a man"; thus "x is a Greek" implies "x is a man", and this holds for all values of x. But the statement is not one of pure mathematics, because it depends upon the particular nature of *Greek* and *man*. We may, however, vary these too, and obtain: If a and b are classes, and a is contained in b, then "x is an a" implies "x is a b". Here at last we have a proposition of pure mathematics, containing three variables and the constants *class*, *contained in* and those involved in the notion of formal implications with variables. So long as any term in our proposition can be turned into a variable, our proposition can be generalized; and so long as this is possible, it is the business of mathematics to do it. If there are several chains of deduction which differ only as to the meaning of the symbols, so that propositions symbolically identical become capable of several interpretations, the proper course, mathematically, is to form the class of meanings which may attach to the symbols, and to assert that the formula in question follows from the hypothesis that the symbols belong to the class in question. In this way, symbols which stood for constants become transformed into variables, and new constants are substituted, consisting of classes to which the old constants belong. Cases of such generalization are so frequent that many will occur at once to every mathematician, and innumerable instances will be given in the present work. Whenever two sets of terms have

* It is necessary to suppose arithmetical addition and multiplication defined (as may be easily done) so that the above formula remains significant when x and y are not numbers.

mutual relations of the same type, the same form of deduction will apply to both. For example, the mutual relations of points in a Euclidean plane are of the same type as those of the complex numbers; hence plane geometry, considered as a branch of pure mathematics, ought not to decide whether its variables are points or complex numbers or some other set of entities having the same type of mutual relations. Speaking generally, we ought to deal, in every branch of mathematics, with any class of entities whose mutual relations are of a specified type; thus the class, as well as the particular term considered, becomes a variable, and the only true constants are the types of relations and what they involve. Now a *type* of relation is to mean, in this discussion, a class of relations characterized by the above formal identity of the deductions possible in regard to the various members of the class; and hence a type of relations, as will appear more fully hereafter, if not already evident, is always a class definable in terms of logical constants.* We may therefore define a type of relation as a class of relations defined by some property definable in terms of logical constants alone.

9. Thus pure mathematics must contain no indefinables except logical constants, and consequently no premisses, or indemonstrable propositions, but such as are concerned exclusively with logical constants and with variables. It is precisely this that distinguishes pure from applied mathematics. In applied mathematics, results which have been shown by pure mathematics to follow from some hypothesis as to the variable are actually asserted of some constant satisfying the hypothesis in question. Thus terms which were variables become constant, and a new premiss is always required, namely: this particular entity satisfies the hypothesis in question. Thus for example Euclidean Geometry, as a branch of pure mathematics, consists wholly of propositions having the hypothesis "*S* is a Euclidean space". If we go on to: "The space that exists is Euclidean", this enables us to assert of the space that exists the consequents of all the hypotheticals constituting Euclidean Geometry, where now the variable *S* is replaced by the constant *actual space*. But by this step we pass from pure to applied mathematics.

10. The connection of mathematics with logic, according to the above account, is exceedingly close. The fact that all mathematical constants are logical constants, and that all the premisses of mathematics are concerned with these, gives, I believe, the precise statement of what philosophers have meant in asserting that mathematics is *à priori*. The fact is that, when once the apparatus of logic has been accepted, all mathematics necessarily follows. The logical constants themselves are to be defined only by enumeration, for they are so fundamental that all the properties by which the class of them might

* One-one, many-one, transitive, symmetrical, are instances of types of relations with which we shall be often concerned.

be defined presuppose some terms of the class. But practically, the method of discovering the logical constants is the analysis of symbolic logic, which will be the business of the following chapters. The distinction of mathematics from logic is very arbitrary, but if a distinction is desired, it may be made as follows. Logic consists of the premisses of mathematics, together with all other propositions which are concerned exclusively with logical constants and with variables but do not fulfil the above definition of mathematics (§ 1). Mathematics consists of all the consequences of the above premisses which assert formal implications containing variables, together with such of the premisses themselves as have these marks. Thus some of the premisses of mathematics, *e.g.* the principle of the syllogism, "if *p* implies *q* and *q* implies r, then *p* implies r", will belong to mathematics, while others, such as "implication is a relation", will belong to logic but not to mathematics. But for the desire to adhere to usage, we might identify mathematics and logic, and define either as the class of propositions containing only variables and logical constants; but respect for tradition leads me rather to adhere to the above distinction, while recognizing that certain propositions belong to both sciences.

From what has now been said, the reader will perceive that the present work has to fulfil two objects, first, to show that all mathematics follows from symbolic logic, and secondly to discover, as far as possible, what are the principles of symbolic logic itself. The first of these objects will be pursued in the following Parts, while the second belongs to Part I. And first of all, as a preliminary to a critical analysis, it will be necessary to give an outline of Symbolic Logic considered simply as a branch of mathematics. This will occupy the following chapter