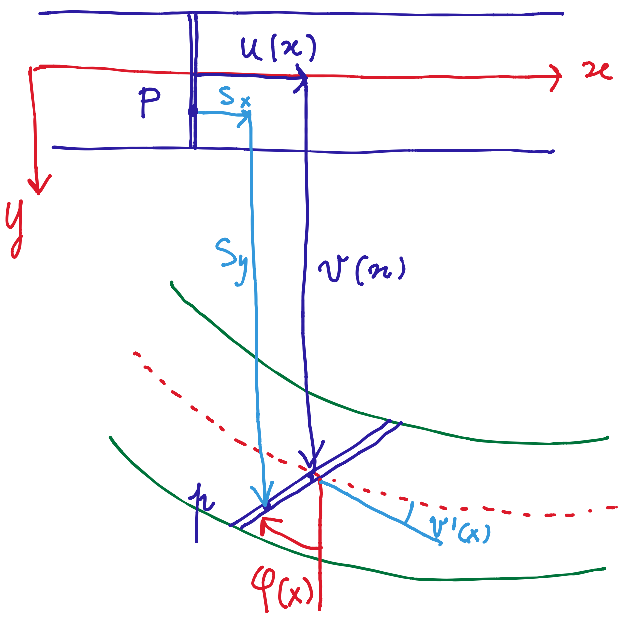
Trave deformabile al taglio (Timoshenko)



$$\varphi(x) \neq v'(x)$$

$$s_x(x,y) = u(x) - \varphi(x)\, y$$

$$s_y(x,y) = v(x)$$

Deformazioni locali ε_x, ε_y, γ_{xy}

$$\varepsilon_x(x,y) = \frac{\partial s_x(x,y)}{\partial x} = \frac{\partial u(x)}{\partial x} - y\frac{\partial \varphi(x)}{\partial x} = u'(x) - y\,\varphi'(x)$$

$$\varepsilon_y(x,y) = \frac{\partial s_y(x,y)}{\partial y} = \frac{\partial v(x)}{\partial y} = 0$$

$$\gamma_{xy}(x,y) = \frac{\partial s_x(x,y)}{\partial y} + \frac{\partial s_y(x,y)}{\partial x} = \boxed{-\varphi(x) + v'(x)}$$

SE $\gamma_{xy} = 0 \Rightarrow$ Eulero-Bernoulli

$$-\varphi(x) + v'(x) = 0$$

$$\varphi(x) = v'(x)$$

DEFORMAZIONI GENERALIZZATE

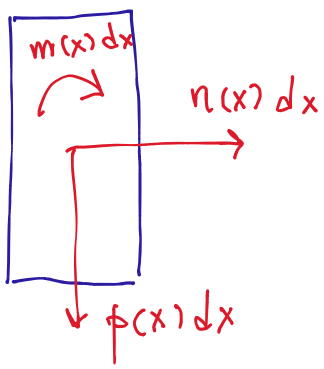
$$\eta(x) = u'(x) \quad ; \quad \chi(x) = -\frac{d\varphi(x)}{dx} \overset{EB}{=} \left(-v''(x)\right)$$

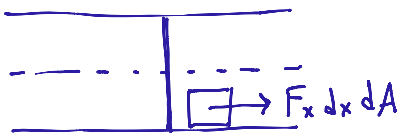
deformazione assiale — curvatura

$$t(x) = v'(x) - \varphi(x)$$

deformazione di taglio

$$\varepsilon_x(x,y) = \eta(x) + y\,\chi(x) \quad ; \quad \gamma_{xy}(x,y) = t(x)$$





$$n(x) = \int_A F_x \, dA$$

$$p(x) = \int_A F_y \, dA$$

$$m(x) = \int_A -y \, F_x \, dA$$

Principio dei lavori virtuali

$$\mathcal{L}_i = \mathcal{L}_e$$

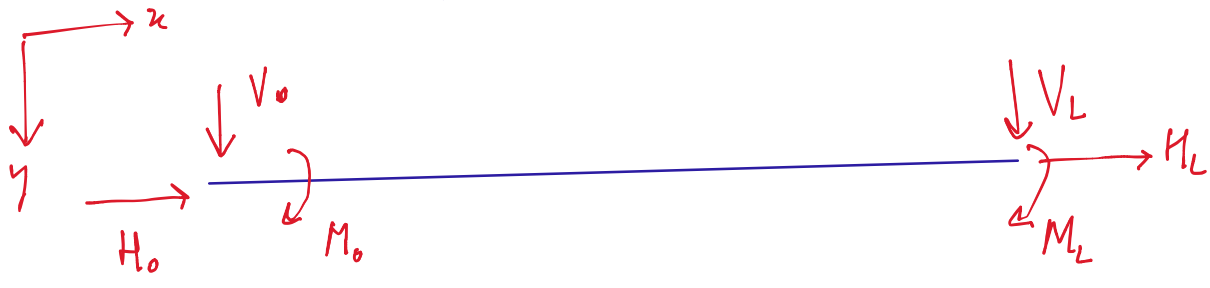
$$\frac{d\mathcal{L}_i}{dx} = \int_A \sigma_{ij}\,\delta\varepsilon_{ij}\,dA = \int_A \left(\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\cancel{\varepsilon_y} + \tau_{xy}\,\delta\gamma_{xy}\right)dA$$

$$0 \Leftarrow \varepsilon_y = 0$$

$$= \int_A \sigma_x\left(\delta\eta(x) + y\,\delta\chi(x)\right)dA + \int_A \tau_{xy}\,\delta t(x)\,dA$$

$$= \delta\eta(x)\underbrace{\int_A \sigma_x\,dA}_{N(x)} + \delta\chi(x)\underbrace{\int_A y\,\sigma_x\,dA}_{M(x)} + \delta t(x)\underbrace{\int_A \tau_{xy}\,dA}_{T(x)}$$

$$\mathcal{L}_i = \int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,dV = \int_0^L \left\{N(x)\,\delta\eta(x) + M(x)\,\delta\chi(x) + T(x)\,\delta t(x)\right\}dx$$



$$\mathcal{L}_e = \int_0^L \{ n(x)\, \delta u(x) + p(x)\, \delta v(x) + m(x)\, \delta \varphi(x) \}\, dx +$$

$$+ H_0\, \delta u(0) + V_0\, \delta v(0) + M_0\, \delta \varphi(0) +$$

$$+ H_L\, \delta u(L) + V_L\, \delta v(L) + M_L\, \delta \varphi(L)$$

$$\mathcal{L}_i = \int_0^L \{ \overset{1}{N(x)\, \delta \eta(x)} + \overset{2}{M(x)\, \delta \chi(x)} + \overset{3}{T(x)\, \delta t(x)} \}\, dx$$

$$\eta(x) \to \delta u'(x) \qquad \chi(x) \to -\delta \varphi'(x) \qquad t(x) \to \delta\left(v'(x) - \varphi(x)\right)$$

INTEGRAZIONI PER PARTI $\displaystyle\int_{x_1}^{x_2} f'(x) g(x)\, dx = f(x) g(x) \Big|_{x_1}^{x_2} - \int f(x) g'(x)\, dx$

1 $$\int_0^L N(x)\, \delta u'(x)\, dx = N(x)\, \delta u(x) \Big|_0^L - \int_0^L N'(x)\, \delta u(x)\, dx$$

2 $$-\int_0^L M(x)\, \delta \varphi'(x)\, dx = -M(x)\, \delta\varphi(x) \Big|_0^L + \int_0^L M'(x)\, \delta \varphi(x)\, dx$$

3 $$\int_0^L T(x)\, \delta\left(v'(x) - \varphi(x)\right) dx = \int_0^L T(x)\, \delta v'(x)\, dx - \int_0^L T(x)\, \delta\varphi(x)\, dx$$

$$= T(x)\, \delta v(x) \Big|_0^L - \int_0^L T'(x)\, \delta v(x)\, dx - \int_0^L T(x)\, \delta \varphi(x)\, dx$$

$$N(x)\, \delta u(x) \Big|_0^L = N(L)\, \delta u(L) - N(0)\, \delta u(0)$$

Applicazione il PLV

$$\left.\begin{aligned}
&N(L)\,\delta u(L) - N(0)\,\delta u(0) - \int_0^L N'(x)\,\delta u(x)\,dx \; + \\
&- M(L)\,\delta\varphi(L) + M(0)\,\delta\varphi(0) + \int_0^L M'(x)\,\delta\varphi(x)\,dx \; + \\
&+ T(L)\,\delta v(L) - T(0)\,\delta v(0) - \int_0^L T'(x)\,\delta v(x)\,dx \; + \\
&- \int_0^L T(x)\,\delta\varphi(x)\,dx
\end{aligned}\right\}\mathcal{L}_i \; =$$

$$\left.\begin{aligned}
&= \int_0^L \left\{ n(x)\,\delta u(x) + p(x)\,\delta v(x) + m(x)\,\delta\varphi(x) \right\} dx \; + \\
&+ H_0\,\delta u(0) + V_0\,\delta v(0) + M_0\,\delta\varphi(0) + H_L\,\delta u(L) + V_L\,\delta v(L) + \\
&\qquad + M_L\,\delta\varphi(L)
\end{aligned}\right\}\mathcal{L}_e$$

$$- \int_0^L \left\{ N'(x) + n(x) \right\} \delta u(x)\,dx - \int_0^L \left\{ T'(x) + p(x) \right\} \delta v(x)\,dx \; +$$

$$+ \int_0^L \left\{ M'(x) - T(x) - m(x) \right\} \delta\varphi(x)\,dx \; +$$

$$+ \left[N(L) - H_L \right] \delta u(L) - \left[N(0) + H_0 \right] \delta u(0) - \left[T(0) + V_0 \right] \delta v(0) \; +$$

$$+ \left[T(L) - V_L \right] \delta v(L) + \left[M(0) - M_0 \right] \delta\varphi(0) - \left[M(L) + M_L \right] \delta\varphi(L) = 0$$

EQUAZIONI D'EQUILIBRIO

$$N'(x) + n(x) = 0 \qquad \begin{matrix} u(0) = \bar{u}_0 \\ u(L) = \bar{u}_L \end{matrix} \quad \text{oppure} \quad \begin{matrix} N(0) = -H_0 \\ N(L) = H_L \end{matrix}$$

$$T'(x) + p(x) = 0 \qquad \begin{matrix} v(0) = \bar{v}_0 \\ v(L) = \bar{v}_L \end{matrix} \quad \text{oppure} \quad \begin{matrix} T(0) = -V_0 \\ T(L) = V_L \end{matrix}$$

$$M'(x) - T(x) - m(x) = 0 \qquad \begin{matrix} \varphi(0) = \bar{\varphi}_0 \\ \varphi(L) = \bar{\varphi}_L \end{matrix} \quad \text{oppure} \quad \begin{matrix} M(0) = M_0 \\ M(L) = -M_L \end{matrix}$$

INTRODUCO IL LEGAME COSTITUTIVO

$$N(x) = EA\,\eta(x) = EA\,u'(x)$$

$$M(x) = EJ\,\chi(x) = -EJ\,\varphi'(x)$$

$$T(x) = GA_t\,t(x) = GA_t\left(v'(x) - \varphi(x)\right)$$

$$\hookrightarrow A_t < A$$

Se la trave è a sezione costante ed omogenea, ALLORA EJ, EA e GA_t sono costanti

- $N'(x) + n(x) = 0$

$$\left[EA\,u'(x)\right]' = -n(x) \implies u''(x) = -\frac{n(x)}{EA}$$

- $T'(x) + p(x) = 0$

$$\left[GA_t\left(v'(x) - \varphi(x)\right)\right]' = -p(x)$$

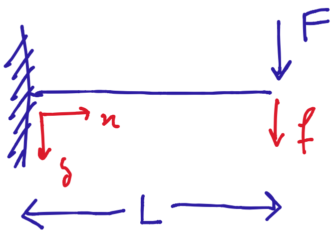
$$\implies v''(x) - \varphi'(x) = -\frac{p(x)}{GA_t}$$

- $M'(x) - T(x) - m(x) = 0$

$$\left[-EJ\,\varphi'(x)\right]' - GA_t\left(v'(x) - \varphi(x)\right) = m(x)$$

$$\implies -EJ\,\varphi''(x) = T(x) + m(x)$$

ESERCIZIO 1



(i) $GA_t\left(v''(x) - \varphi'(x)\right) = -p(x)$

(ii) $-EJ\,\varphi''(x) = T(x) + m(x)$

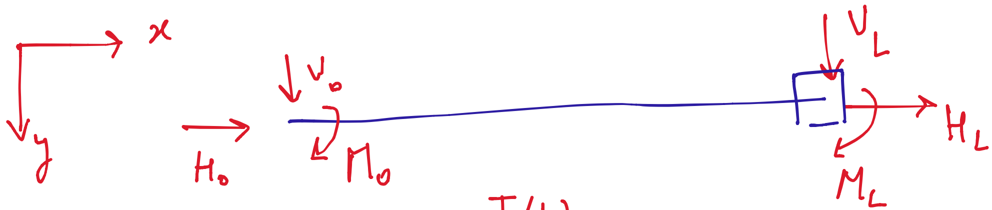
(i) $\boxed{v''(x) - \varphi'(x) = 0}$

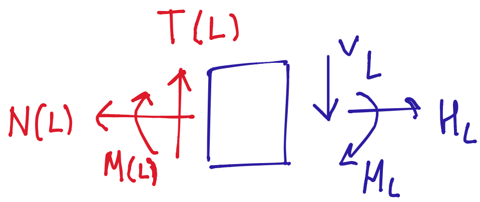
(ii) $-EJ\,\varphi''(x) - T(x) = 0$

$$\boxed{-EJ\,\varphi''(x) - GA_t\left(v'(x) - \varphi(x)\right) = 0}$$

CONDIZIONI AL BORDO

1 $v(0) = 0$ 2 $\varphi(0) = 0$





$T(L) = V_L$ e $M(L) = -M_L$

3 $V_L = F$ 4 $M_L = 0 \Rightarrow EJ\,\varphi'(L) = 0$

$\Rightarrow \boxed{\varphi'(L) = 0}$

$\downarrow$

$T(L) = \boxed{GA_t\left(v'(L) - \varphi(L)\right) = F}$

① $v(0)=0$ ② $\varphi(0)=0$

③ $\varphi'(L)=0$ ④ $GA_t\left[v'(L)-\varphi(L)\right]=F$

(i) $GA_t\left(v''(x)-\varphi'(x)\right)=0$

(ii) $EJ\,\varphi''(x)+GA_t\left(v'(x)-\varphi(x)\right)=0$

Integrando la (i)

$$\underbrace{GA_t\left(v'(x)-\varphi(x)\right)}_{T(x)}=C$$

$T(L)=F \Rightarrow C=F$

Dalla (ii)

$$EJ\,\varphi''(x)+F=0$$

$$\varphi''(x)=-\frac{F}{EJ}$$

$$\varphi'(x)=-\frac{F}{EJ}x+C_1$$

$$\varphi(x)=-\frac{1}{2}\frac{F}{EJ}x^2+C_1x+C_2$$

② $\varphi(0) = 0$ ③ $\varphi'(L) = 0$

$$\varphi(0) = \boxed{C_2 = 0}$$

$$\varphi'(L) = -\frac{FL}{EJ} + C_1 = 0 \Rightarrow \boxed{C_1 = \frac{FL}{EJ}}$$

$$\varphi(x) = -\frac{F}{2EJ}x^2 + \frac{FL}{EJ}x = \frac{Fx}{EJ}\left(L - \frac{x}{2}\right)$$

Ritorniamo al taglio

$$GA_t\left(v'(x) - \varphi(x)\right) = F\ ;$$

$$v'(x) - \varphi(x) = \frac{F}{GA_t}\ ;$$

$$v'(x) = \varphi(x) + \frac{F}{GA_t}$$

$$v'(x) = \frac{F}{GA_t} + \frac{Fx}{EJ}\left(L - \frac{x}{2}\right)$$

$$v(x) = \frac{F}{GA_t}x + \frac{FL}{2EJ}x^2 - \frac{F}{6EJ}x^3 + D$$

① $v(0)=0$

$$v(0)=D=0 \Rightarrow \boxed{D=0}$$

$$v(x)=\frac{Fx}{GA_t}+\frac{Fx^2}{2EJ}\left(L-\frac{x}{3}\right)$$

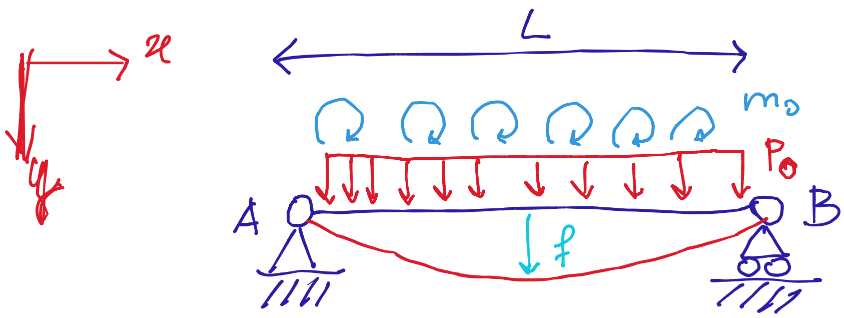
$$\varphi(x)=\frac{Fx}{EJ}\left(L-\frac{x}{2}\right)$$

$$f=v(x=L)=\frac{FL}{GA_t}+\frac{FL^2}{\cancel{2}EJ}\,\frac{\cancel{2}L}{3}$$

$$=\frac{FL}{GA_t}+\frac{FL^3}{3EJ}$$

freccia dovuta al taglio

freccia dovuta a flessione



x
y
L
m_0
P_0
A
B
f