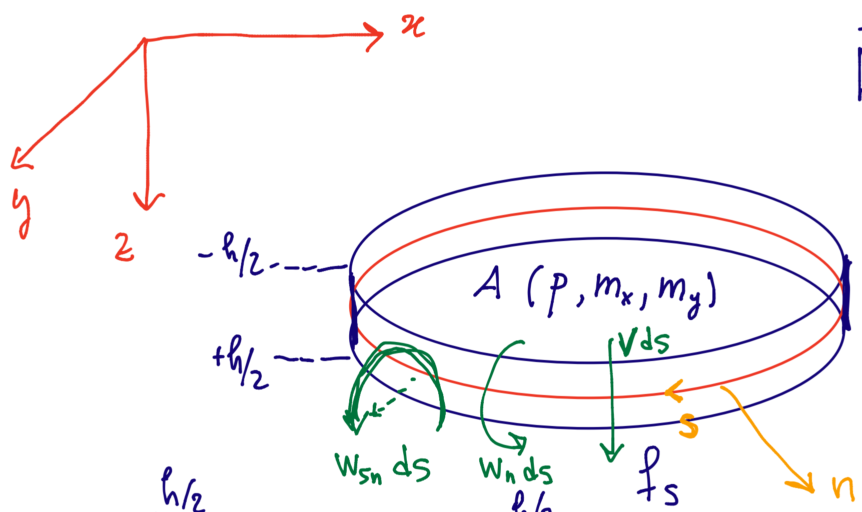
PIASTRA DI REISSNER-MINDLIN : PROBLEMA FLESSIONALE



$$\vec{F} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

$$f_x, f_y \rightarrow f_n, f_s$$

$$V = \int_{-h/2}^{h/2} f_z\, dz \quad ; \quad W_n = \int_{-h/2}^{h/2} -z\, f_n\, dz \quad ; \quad W_{sn} = \int_{-h/2}^{h/2} -z\, f_s\, dz$$

$$\mathcal{L}_e = \int_A (p\,\delta w + m_x\,\delta\varphi_x + m_y\,\delta\varphi_y)\, dA + \int_\Gamma (V\,\delta w + W_n\,\delta\varphi_n + W_{sn}\,\delta\varphi_s)\, ds$$

$$\mathcal{L}_e = \int_A \left(\vec{P}_m^{\,t}\,\delta\vec{u}_m + \boxed{\vec{P}_f^{\,t}\,\delta\vec{u}_f}\right) dA$$

$$\mathcal{L}_i = \int_A \left(\vec{Q}_m^{\,t}\,\delta\vec{q}_m + \vec{Q}_f^{\,t}\,\delta\vec{q}_f\right) dA$$

$$\vec{P}_m^{\,t} = (n_x, m_y) \qquad \vec{P}_f^{\,t} = (p, m_x, m_y)$$

$$\vec{U}_m^{\,t} = (u, v) \qquad \vec{U}_f^{\,t} = (w, \varphi_x, \varphi_y)$$

$$\vec{Q}_m^{\,t} = (N_x, N_y, N_{xy}) \qquad \vec{Q}_f^{\,t} = (M_x, M_y, M_{xy}, T_{xz}, T_{yz})$$

$$\vec{q}_m^{\,t} = (\eta_x, \eta_y, \eta_{xy}) \qquad \vec{q}_f^{\,t} = (\chi_x, \chi_y, \chi_{xy}, t_x, t_y)$$

$$\eta_x = \frac{\partial u}{\partial x} \quad ; \quad \eta_y = \frac{\partial v}{\partial y} \quad ; \quad \eta_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\chi_x = -\frac{\partial \varphi_x}{\partial x} \quad ; \quad \chi_y = -\frac{\partial \varphi_y}{\partial y} \quad ; \quad \chi_{xy} = -\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right)$$

$$t_x = \frac{\partial w}{\partial x} - \varphi_x \quad ; \quad t_y = \frac{\partial w}{\partial y} - \varphi_y$$

LAVORO INTERNO

$$\mathcal{L}_i = \int_A \left(M_x \delta\chi_x + M_y \delta\chi_y + M_{xy} \delta\chi_{xy} + T_x \delta t_x + T_y \delta t_y\right) dA$$

$$= \int_A \left[M_x \left(-\frac{\partial \delta\varphi_x}{\partial x}\right) + M_y \left(-\frac{\partial \delta\varphi_y}{\partial y}\right) + M_{xy} \left(-\frac{\partial \delta\varphi_x}{\partial y} - \frac{\partial \delta\varphi_y}{\partial x}\right) + \right.$$

$$+ T_x\left(\frac{\partial \delta w}{\partial x} - \delta\varphi_x\right) + T_y\left(\frac{\partial \delta w}{\partial y} - \delta\varphi_y\right)\Bigg] dA$$

$$= \int_A -\left[M_x \frac{\partial \delta\varphi_x}{\partial x} + M_y \frac{\partial \delta\varphi_y}{\partial y} + M_{xy}\left(\frac{\partial \delta\varphi_x}{\partial y} + \frac{\partial \delta\varphi_y}{\partial x}\right) - T_x \frac{\partial \delta w}{\partial x} - T_y \frac{\partial \delta w}{\partial y} + \right.$$

$$\left. + T_x \delta\varphi_x + T_y \delta\varphi_y \right] dA$$

$$D\left[f(x)\, g(x)\right] = f'(x)\, g(x) + f(x)\, g'(x)$$

LEIBNITZ' RULE

$$\Downarrow$$

$$f'(x)\, g(x) = D\left[f(x)\, g(x)\right] - f(x)\, g'(x)$$

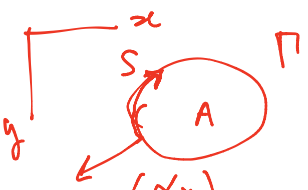
$$\mathcal{L}_i = \int_A -\left[\frac{\partial}{\partial x}\left(M_x \delta\varphi_x\right) + \frac{\partial}{\partial y}\left(M_y \delta\varphi_y\right) + \frac{\partial}{\partial y}\left(M_{xy}\delta\varphi_x\right) + \frac{\partial}{\partial x}\left(M_{xy}\delta\varphi_y\right) + \right.$$

$$\left. - \frac{\partial}{\partial x}\left(T_x \delta w\right) - \frac{\partial}{\partial y}\left(T_y \delta w\right) + T_x \delta\varphi_x + T_y \delta\varphi_y \right] dA +$$

$$- \int_A -\left(\frac{\partial M_x}{\partial x}\delta\varphi_x + \frac{\partial M_y}{\partial y}\delta\varphi_y + \frac{\partial M_{xy}}{\partial y}\delta\varphi_x + \frac{\partial M_{xy}}{\partial x}\delta\varphi_y + \right.$$

$$\left. - \frac{\partial T_x}{\partial x}\delta w - \frac{\partial T_y}{\partial y}\delta w \right] dA$$

T. DELLA DIVERGENZA



$$n = \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix}$$

$$\vec{g} = \begin{pmatrix} g_x \\ g_y \end{pmatrix}$$

$$\int_A \left(\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y}\right) dA = \int_\Gamma \left(g_x \alpha_x + g_y \alpha_y\right) ds$$

$$\mathcal{L}_i = \int_A \frac{\partial}{\partial x}\underbrace{\left(T_x \delta w - M_x \delta\varphi_x - M_{xy}\delta\varphi_y\right)}_{\partial x} + \frac{\partial}{\partial y}\underbrace{\left(T_y \delta w - M_{xy}\delta\varphi_x - M_y \delta\varphi_y\right)}_{\partial y} dA$$

$$\boxed{+\int_A \left[\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - T_x\right)\delta\varphi_x + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - T_y\right)\delta\varphi_y + - \left(\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y}\right)\delta w\right] dA} \quad \mathcal{J}$$

Applico il teorema delle divergenze

$$\mathcal{L}_i = \int_\Gamma \left(T_x \alpha_x \delta w - M_x \alpha_x \delta\varphi_x - M_{xy}\alpha_x \delta\varphi_y + T_y \alpha_y \delta w - M_{xy}\alpha_y \delta\varphi_x + - M_y \alpha_y \delta\varphi_y\right) ds + \mathcal{J}$$

$$= \int_\Gamma \left[\left(T_x \alpha_x + T_y \alpha_y\right)\delta w - \left(M_x \alpha_x + M_{xy}\alpha_y\right)\delta\varphi_x + - \left(M_{xy}\alpha_x + M_y \alpha_y\right)\delta\varphi_y\right] ds + \mathcal{J}$$

Invochiamo il PLV

$$\mathcal{L}_i = \mathcal{L}_e \quad \Rightarrow \quad \mathcal{L}_i - \mathcal{L}_e = 0$$

$$\mathcal{L}_e = \int_A \left(p\,\delta w + m_x \delta\varphi_x + m_y \delta\varphi_y\right) dA + \int_\Gamma \left(V \delta w + W_n \delta\varphi_n + W_{sn}\delta\varphi_s\right) ds$$

$$\int_A \left[\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - T_x - m_x \right) \delta\varphi_x + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - T_y - m_y \right) \delta\varphi_y + \right.$$

$$\left. - \left(\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + p \right) \delta w \right] dA +$$

$$\int_\Gamma \left[\left(T_x \alpha_x + T_y \alpha_y - V \right) \delta w - \left(M_x \alpha_x + M_{xy} \alpha_y \right) \underline{\delta\varphi_x} + \right.$$

$$\left. - \left(M_{xy} \alpha_x + M_y \alpha_y \right) \underline{\delta\varphi_y} - W_n \underline{\delta\varphi_n} - W_{sn} \underline{\delta\varphi_s} \right] dS = 0$$

EQUAZIONI D'EQUILIBRIO

φ_n e φ_s sono esprimibili in termini di φ_x e φ_y

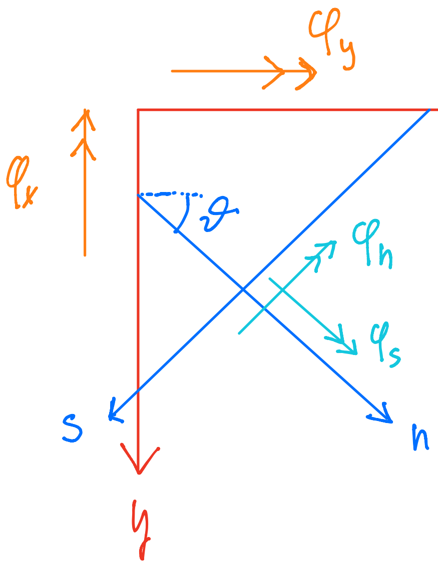
$$\begin{cases} \dfrac{\partial M_x}{\partial x} + \dfrac{\partial M_{xy}}{\partial y} - T_x = m_x \\[2ex] \dfrac{\partial M_{xy}}{\partial x} + \dfrac{\partial M_y}{\partial y} - T_y = m_y \\[2ex] \dfrac{\partial T_x}{\partial x} + \dfrac{\partial T_y}{\partial y} = -p \end{cases}$$

equilibrio alla rotazione intorno a x e a y

equilibrio alla traslazione

CONDIZIONI AL BORDO

$\delta w = 0 \quad (w = \bar{w})$ oppure $T_x \alpha_x + T_y \alpha_y = V$



$$\vec{n} = \begin{Bmatrix} \alpha_x = \cos\vartheta \\ \alpha_y = \sin\vartheta \end{Bmatrix}$$

$$\delta\varphi_x = \delta\varphi_n \alpha_x - \delta\varphi_s \alpha_y$$

$$\delta\varphi_y = \delta\varphi_n \alpha_y + \delta\varphi_s \alpha_x$$

$$\begin{cases} \varphi_x = \varphi_n \cos\vartheta - \varphi_s \sin\vartheta \\ \varphi_y = \varphi_n \sin\vartheta + \varphi_s \cos\vartheta \end{cases} \qquad \begin{cases} \varphi_s = \varphi_y \cos\vartheta - \varphi_x \sin\vartheta \\ \varphi_n = \varphi_x \cos\vartheta + \varphi_y \sin\vartheta \end{cases}$$

$$\underline{R} = \begin{pmatrix} \alpha_x & -\alpha_y \\ \alpha_y & \alpha_x \end{pmatrix} = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$

$$\begin{pmatrix} \varphi_s \\ \varphi_n \end{pmatrix} = \underline{R} \begin{pmatrix} \varphi_x \\ \varphi_y \end{pmatrix} \quad ; \quad \begin{pmatrix} \varphi_x \\ \varphi_y \end{pmatrix} = \underline{R}^{+} \begin{pmatrix} \varphi_s \\ \varphi_n \end{pmatrix}$$

$$\boxed{\underline{R}\,\underline{R}^{+} = \underline{R}^{+}\underline{R} = \mathbb{1}}$$

$$\int_{\Gamma}\Big[\big(T_x\alpha_x+T_y\alpha_y-V\big)\delta W-\big(M_x\alpha_x+M_{xy}\alpha_y\big)\delta\varphi_x+$$
$$-\big(M_{xy}\alpha_x+M_y\alpha_y\big)\delta\varphi_y-W_n\,\delta\varphi_n-W_{sn}\,\delta\varphi_s\Big]dS=0$$

$$\begin{cases}\varphi_x=\varphi_n\,\alpha_x-\varphi_s\,\alpha_y\\ \varphi_y=\varphi_n\,\alpha_y+\varphi_s\,\alpha_x\end{cases}$$

$$\int_{\Gamma}\Big[\big(V-T_x\alpha_x-T_y\alpha_y\big)\delta W+\big(M_x\alpha_x+M_{xy}\alpha_y\big)\big(\delta\varphi_n\alpha_x-\delta\varphi_s\alpha_y\big)+$$
$$+\big(M_{xy}\alpha_x+M_y\alpha_y\big)\big(\delta\varphi_n\alpha_y+\delta\varphi_s\alpha_x\big)+W_n\,\delta\varphi_n$$
$$+W_{sn}\,\delta\varphi_s\Big]dS$$

$$=\int_{\Gamma}\Big[\big(V-T_x\alpha_x-T_y\alpha_y\big)\delta w+M_x\alpha_x^2\,\delta\varphi_n+M_{xy}\alpha_x\alpha_y\,\delta\varphi_n-M_x\alpha_x\alpha_y\,\delta\varphi_s+$$
$$-M_{xy}\alpha_y^2\,\delta\varphi_s+M_{xy}\alpha_x\alpha_y\,\delta\varphi_n+M_{xy}\alpha_x^2\,\delta\varphi_s+M_y\alpha_y^2\,\delta\varphi_n+$$
$$+M_y\alpha_x\alpha_y\,\delta\varphi_s+W_n\delta\varphi_n+W_{sn}\delta\varphi_s\Big]dS$$

$$=\int_{\Gamma}\Big\{\Big[\big(V-T_x\alpha_x-T_y\alpha_y\big)\delta w+\big(W_n+M_x\alpha_x^2+M_y\alpha_y^2+2M_{xy}\alpha_x\alpha_y\big)\delta\varphi_n+$$
$$+\Big[W_{sn}+\big(M_y-M_x\big)\alpha_x\alpha_y+M_{xy}\big(\alpha_x^2-\alpha_y^2\big)\Big]\delta\varphi_s\Big\}dS=0$$

CONDIZIONI AL BORDO

$$\delta w = 0 \quad (w = \bar{w})$$
$$\delta \varphi_n = 0 \quad (\varphi_n = \bar{\varphi}_n)$$
$$\delta \varphi_s = 0 \quad (\varphi_s = \bar{\varphi}_s)$$

oppure

$$\begin{cases} V - T_x \alpha_x - T_y \alpha_y = 0 \\ W_n + \underline{M_x \alpha_x^2 + M_y \alpha_y^2 + 2 M_{xy} \alpha_x \alpha_y} = 0 \\ W_{sn} + \underline{(M_y - M_x)\alpha_x \alpha_y + M_{xy}(\alpha_x^2 - \alpha_y^2)} = 0 \end{cases}$$

INTERPRETIAMO

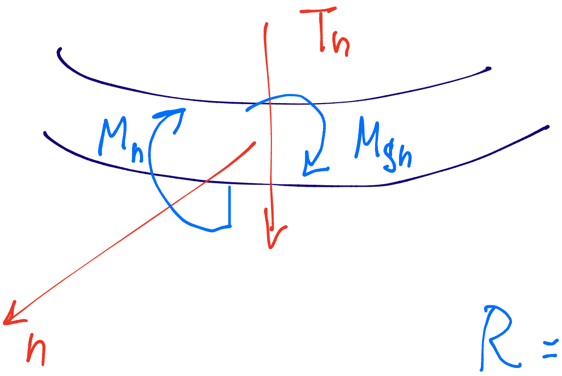
$$V = T_x \alpha_x + T_y \alpha_y$$

$$\begin{cases} \varphi_s = \varphi_y \cos\vartheta - \varphi_x \sin\vartheta \\ \varphi_n = \varphi_x \cos\vartheta + \varphi_y \sin\vartheta \end{cases}$$

$$T_x \alpha_x + T_y \alpha_y = T_n$$
$$-T_x \alpha_y + T_y \alpha_x = T_s$$

$$\begin{cases} g_s = g_y \alpha_x - g_x \alpha_y \\ g_n = g_x \alpha_x + g_y \alpha_y \end{cases}$$

$$V = T_n$$



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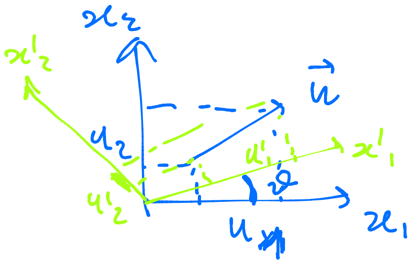


$$\underline{R}\,\vec{u} = \vec{u}'$$

$$\underline{R} = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$

$$\underline{R}^T \underline{M}\, \underline{R} = \underline{M}'$$

$$\underline{R}^t \begin{pmatrix} M_x & M_{xy} \\ M_{xy} & M_y \end{pmatrix} \underline{R} = \begin{pmatrix} M_n & M_{sn} \\ M_{sn} & M_s \end{pmatrix}$$

$$\underline{R} = \begin{pmatrix} \alpha_x & -\alpha_y \\ \alpha_y & \alpha_x \end{pmatrix} \qquad \underline{R}^t = \begin{pmatrix} \alpha_x & \alpha_y \\ -\alpha_y & \alpha_x \end{pmatrix}$$

$$\underline{R}^t \, \underline{M} \, \underline{R} = \begin{pmatrix} \alpha_x & \alpha_y \\ -\alpha_y & \alpha_x \end{pmatrix} \begin{pmatrix} M_x & M_{xy} \\ M_{xy} & M_y \end{pmatrix} \begin{pmatrix} \alpha_x & -\alpha_y \\ \alpha_y & \alpha_x \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_x & \alpha_y \\ -\alpha_y & \alpha_x \end{pmatrix} \begin{pmatrix} M_x \alpha_x + M_{xy} \alpha_y & -\alpha_y M_x + \alpha_x M_{xy} \\ M_{xy} \alpha_x + M_y \alpha_y & -M_{xy} \alpha_y + M_y \alpha_x \end{pmatrix}$$

$$= \begin{pmatrix} M_x \alpha_x^2 + M_{xy} \alpha_x \alpha_y + M_{xy} \alpha_x \alpha_y + M_y \alpha_y^2 & \bullet \\ -M_x \alpha_x \alpha_y - M_{xy} \alpha_y^2 + M_{xy} \alpha_x^2 + M_y \alpha_x \alpha_y & \bullet \end{pmatrix}$$

$$= \begin{pmatrix} M_x \alpha_x^2 + M_y \alpha_y^2 + 2 M_{xy} \alpha_x \alpha_y = M_n & \bullet \\ (M_y - M_x) \alpha_x \alpha_y + M_{xy} (\alpha_x^2 - \alpha_y^2) = M_{sn} & \bullet \end{pmatrix}$$

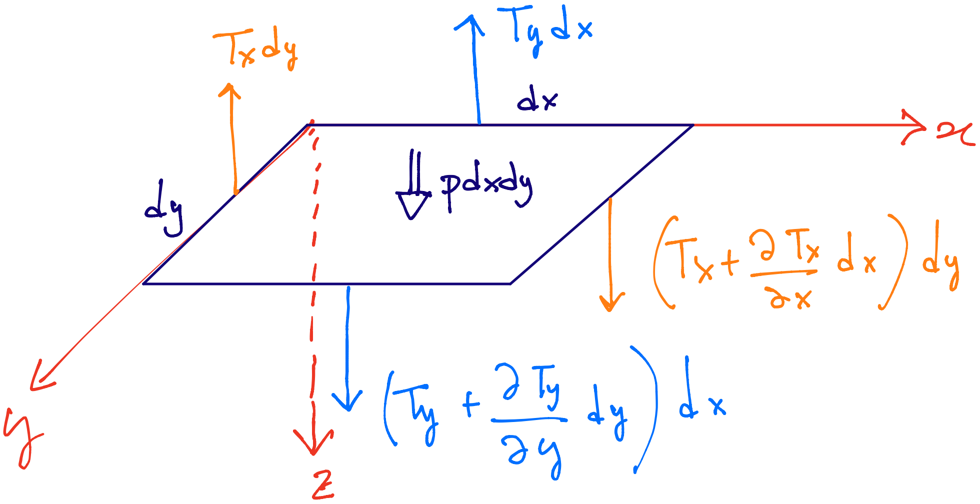
QUINDI LE CONDIZIONI AL BORDO SONO:

$$W = \overline{W} \qquad\qquad T_n = V$$

$$\varphi_n = \overline{\varphi}_n \qquad \text{oppure} \qquad M_n = -W_n$$

$$\varphi_s = \overline{\varphi}_s \qquad\qquad M_{sn} = -W_{sn}$$

EQUAZIONI DI EQUILIBRIO OTTENUTE COL BILANCIO



$$0 = p\,dx\,dy - T_y\,dx + \left(T_y + \frac{\partial T_y}{\partial y}dy\right)dx - T_x\,dy + \left(T_x + \frac{\partial T_x}{\partial x}dx\right)dy$$

$$p\,dx\,dy - \cancel{T_y\,dx} + \cancel{T_y\,dx} + \frac{\partial T_y}{\partial y}dx\,dy - \cancel{T_x\,dy} + \cancel{T_x\,dy} + \frac{\partial T_x}{\partial x}dx\,dy = 0$$

$$p + \frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} = 0$$

A PAG 159 Fig 9.8 $\rightarrow$ equilibrio e rotazione