LEGAME COSTITUTIVO

$$\vec{q} = \begin{pmatrix} \vec{q}_m \\ \vec{q}_f \end{pmatrix} \qquad \vec{q}_m = \begin{pmatrix} \eta_x \\ \eta_y \\ \eta_{xy} \end{pmatrix} \qquad \vec{q}_f = \begin{pmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \\ t_x \\ t_y \end{pmatrix}$$

variabili cinematiche

$$\vec{Q} = \begin{pmatrix} \vec{Q}_m \\ \vec{Q}_f \end{pmatrix} \qquad \vec{Q}_m = \begin{pmatrix} N_x \\ N_y \\ N_{xy} \end{pmatrix} \qquad \vec{Q}_f = \begin{pmatrix} M_x \\ M_y \\ M_{xy} \\ T_x \\ T_y \end{pmatrix}$$

variabili statiche

SE IL PROBLEMA E' ELASTICO

$$\vec{Q} = \mathbb{D} \vec{q}$$

$$\vec{\sigma} = \begin{pmatrix} \vec{\sigma}_p \\ \vec{\tau}_z \end{pmatrix} \qquad \vec{\sigma}_p = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \qquad \vec{\tau}_z = \begin{pmatrix} \tau_{xz} \\ \tau_{yz} \end{pmatrix}$$

$$\vec{\varepsilon} = \begin{pmatrix} \vec{\varepsilon}_p \\ \vec{\gamma}_z \end{pmatrix} \qquad \vec{\varepsilon}_p = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \qquad \vec{\gamma}_z = \begin{pmatrix} \gamma_{xz} \\ \gamma_{yz} \end{pmatrix}$$

$$\vec{\sigma}_p = \underline{d}_p \vec{\varepsilon}_p \qquad \mathbb{D}$$

$$\vec{\tau}_z = \underline{d}_z \vec{\gamma}_z$$

$$\vec{\varepsilon}_p = \vec{\eta} + z\vec{\chi} \rightarrow \vec{\eta} = \vec{q}_m \qquad \vec{\chi} = \begin{pmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{pmatrix}$$

$$\vec{\gamma}_z = \vec{t} \qquad \vec{t} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

$$\vec{M} = \begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} \qquad \vec{T} = \begin{pmatrix} T_x \\ T_y \end{pmatrix}$$

$$\vec{N} = \int_{-h/2}^{h/2} \vec{\sigma}_p \, dz \qquad N_x = \int_{-h/2}^{h/2} \sigma_x \, dz$$

$N_y = \dots$

$N_{xy} = \dots$

$$\vec{M} = \int_{-h/2}^{h/2} z\, \vec{\sigma}_p \, dz \qquad M_x = \int_{-h/2}^{h/2} z\, \sigma_x \, dz$$

$M_y \dots$

$M_{xy} \dots$

$$\vec{T} = \int_{-h/2}^{h/2} \vec{\tau}_z \, dz \qquad T_x = \int_{-h/2}^{h/2} \tau_{xz} \, dz$$

$T_y \dots$

Applico il legame costitutivo

$$\vec{N} = \int_{-h/2}^{h/2} \vec{\sigma}_p \, dz = \int_{-h/2}^{h/2} \underline{d}_p \, \vec{\varepsilon}_p \, dz = \int_{-h/2}^{h/2} \underline{d}_p \vec{\eta} \, dz + \int_{-h/2}^{h/2} z \underline{d}_p \vec{\chi} \, dz \qquad z^0 \quad z^1$$

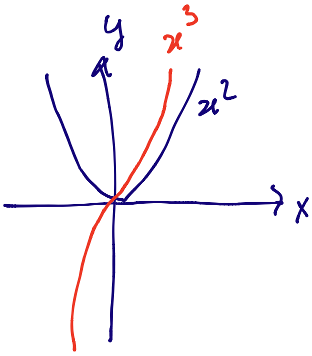
$$\vec{M} = \int_{-h/2}^{h/2} z\, \vec{\sigma}_p \, dz = \int_{-h/2}^{h/2} z\, \underline{d}_p \, \vec{\varepsilon}_p \, dz = \int_{-h/2}^{h/2} z\, \underline{d}_p \vec{\eta} \, dz + \int_{-h/2}^{h/2} z^2 \underline{d}_p \vec{\chi} \, dz$$

$$\vec{T} = \int_{-h/2}^{h/2} \vec{T}_z \, dz = \int_{-h/2}^{h/2} \underline{d}_z \vec{\gamma}_z \, dz = \int_{-h/2}^{h/2} \underline{d}_z \, \vec{t} \, dz$$

$$\int_{-h/2}^{h/2} z \, dz = 0 \qquad \int_{-h/2}^{h/2} f(z) \, dz = 0$$

$$\Downarrow$$

$f(z)$ è dispari



dispari $\quad f(-z) = -f(z)$

pari $\quad f(-z) = f(z)$

QUINDI

$$\vec{N} = \int_{-h/2}^{h/2} \underline{d}_p \, dz \, \vec{\eta} \; ; \quad \vec{M} = \int_{-h/2}^{h/2} z^2 \underline{d}_p \, dz \, \vec{\chi} \; ; \quad \vec{T} = \int_{-h/2}^{h/2} \underline{d}_z \, dz \, \vec{t}$$

perché $\vec{\eta}$, $\vec{\chi}$ e $\vec{t}$ non dipendono da z

Per una piastra omogenea e isotropa

$$G = \frac{E}{2(1+\nu)}$$

$$\underline{d}_p = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \; ; \quad \underline{d}_z = \frac{5}{6} G \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\,\varepsilon_y\right) \quad ; \quad \sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\,\varepsilon_x\right)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\,\gamma_{xy} = G\,\gamma_{xy}$$

$$\int_{-h/2}^{h/2} dz = z\Big|_{-h/2}^{h/2} = h \quad ; \quad \int_{-h/2}^{h/2} z^2\,dz = \frac{z^3}{3}\Big|_{-h/2}^{h/2} = \frac{h^3}{12}$$

$$\vec{N} = h\,\underline{d}_p\,\vec{\eta} \quad ; \quad \vec{M} = \frac{h^3}{12}\,\underline{d}p\,\vec{\chi} \quad ; \quad \vec{T} = h\,\underline{d}_z\,\vec{t}$$

$$N_x = \frac{hE}{1-\nu^2}\left(\eta_x + \nu\,\eta_y\right)$$

$$N_y = \frac{hE}{1-\nu^2}\left(\eta_y + \nu\,\eta_x\right)$$

$$N_{xy} = \frac{hE}{1-\nu^2}\cdot\frac{1-\nu}{2}\,\eta_{xy} = \frac{hE}{2(1+\nu)}\,\eta_{xy}$$

$$\frac{h^3 E}{12(1-\nu^2)} = D$$

$$M_x = D(\chi_x + \nu \chi_y) \; ; \; M_y = D(\chi_y + \nu \chi_x)$$

$$M_{xy} = D \frac{1-\nu}{2} \chi_{xy}$$

$$\begin{pmatrix} T_x \\ T_y \end{pmatrix} = \underbrace{\frac{5}{6} G h}_{Z} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_x \\ t_y \end{pmatrix} = \begin{pmatrix} Z t_x \\ Z t_y \end{pmatrix}$$

Il problema flessionale DOPO il legame costitutivo

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - T_x = m_x \quad (1)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - T_y = m_y \quad (2)$$

$$\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} = -p \quad (3)$$

$$\chi_x = -\frac{\partial \varphi_x}{\partial x} \quad ; \quad \chi_y = -\frac{\partial \varphi_y}{\partial y} \quad ; \quad \chi_{xy} = -\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right)$$

$$t_x = \frac{\partial w}{\partial x} - \varphi_x \qquad t_y = \frac{\partial w}{\partial y} - \varphi_y$$

$$(A) \quad M_x = D\left(\chi_x + \nu \chi_y\right) = -D\left(\frac{\partial \varphi_x}{\partial x} + \nu \frac{\partial \varphi_y}{\partial y}\right)$$

$$(B) \quad M_y = D\left(\chi_y + \nu \chi_x\right) = -D\left(\frac{\partial \varphi_y}{\partial y} + \nu \frac{\partial \varphi_x}{\partial x}\right)$$

(c) $M_{xy} = D\,\frac{1-\nu}{2}\,\chi_{xy} = -D\,\frac{1-\nu}{2}\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right)$

DERIVO

(1) rispetto a x

$$\frac{\partial}{\partial x}\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - T_x\right) = \frac{\partial m_x}{\partial x}$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} - \frac{\partial T_x}{\partial x} = \frac{\partial m_x}{\partial x}$$

(2) rispetto a y

$$\frac{\partial}{\partial y}\left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - T_y\right) = \frac{\partial m_y}{\partial y}$$

$$\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial T_y}{\partial y} = \frac{\partial m_y}{\partial y}$$

SOMMO MEMBRO A MEMBRO

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} - \underbrace{\left(\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y}\right)}_{-p \text{ per la } (3)} = \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y}$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = -p + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y}$$

A $$\frac{\partial^2 M_x}{\partial x^2} = -D\left(\frac{\partial^3 \varphi_x}{\partial x^3} + \nu \frac{\partial^3 \varphi_y}{\partial x^2 \partial y}\right)$$

B $$\frac{\partial^2 M_y}{\partial y^2} = -D\left(\frac{\partial^3 \varphi_y}{\partial y^3} + \nu \frac{\partial^3 \varphi_x}{\partial x \partial y^2}\right)$$

C $$\frac{\partial^2 M_{xy}}{\partial x \partial y} = -D\,\frac{1-\nu}{2}\left(\frac{\partial^3 \varphi_x}{\partial x \partial y^2} + \frac{\partial^3 \varphi_y}{\partial x^2 \partial y}\right)$$

SOMMO TUTTO (con $2\frac{\partial^2 M_{xy}}{\partial x \partial y}$)

$$-D\left[\frac{\partial^3 \varphi_x}{\partial x^3}+\nu\cancel{\frac{\partial^3 \varphi_y}{\partial x^2 \partial y}}+\frac{\partial^3 \varphi_y}{\partial y^3}+\nu\cancel{\frac{\partial^3 \varphi_x}{\partial x \partial y^2}}+(1-\cancel{\nu})\left(\cancel{\frac{\partial^3 \varphi_x}{\partial x \partial y^2}}+\cancel{\frac{\partial^3 \varphi_y}{\partial x^2 \partial y}}\right)\right]=$$

$$=-p+\frac{\partial m_x}{\partial x}+\frac{\partial m_y}{\partial y}$$

$$-D\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial \varphi_x}{\partial x}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial \varphi_y}{\partial y}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial \varphi_y}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial \varphi_x}{\partial x}\right)\right]=$$

$$=-p+\frac{\partial m_x}{\partial x}+\frac{\partial m_y}{\partial y}$$

$$\mathcal{M}=D\left(\frac{\partial \varphi_x}{\partial x}+\frac{\partial \varphi_y}{\partial y}\right)$$

$$\Delta=\nabla^2$$

$$\nabla^2(\cdot)=\frac{\partial^2}{\partial x^2}(\cdot)+\frac{\partial^2}{\partial y^2}(\cdot)$$

$$\nabla^2\mathcal{M}=p-\left(\frac{\partial m_x}{\partial x}+\frac{\partial m_y}{\partial y}\right)$$

prima equazione di Mindlin

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per ricavare le restanti equazioni consideriamo le componenti di taglio

$$T_x=Z\,t_x=Z\left(\frac{\partial w}{\partial x}-\varphi_x\right)$$

$$T_y=Z\,t_y=Z\left(\frac{\partial w}{\partial y}-\varphi_y\right)$$

③

$$\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} = -p$$

$$Z\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi_x}{\partial x} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial \varphi_y}{\partial y}\right) = -p$$

$$Z\underbrace{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)}_{\nabla^2 w} - Z\underbrace{\left(\frac{\partial \varphi_x}{\partial x} + \frac{\partial \varphi_y}{\partial y}\right)}_{\frac{M_0}{D}} = -p$$

$$\nabla^2 w - \frac{M_0}{D} = -\frac{p}{Z}$$

seconda equazione di Mindlin

$$\nabla^2\left(\frac{\partial \varphi_x}{\partial y} - \frac{\partial \varphi_y}{\partial x}\right) = \frac{Z}{D\left(\frac{1-\nu}{2}\right)}\left(\frac{\partial \varphi_x}{\partial y} - \frac{\partial \varphi_y}{\partial x}\right)$$

terza equazione di Mindlin

Proviamo a ricavare la terza equazione

(i) $T_x = Z\left(\frac{\partial w}{\partial x} - \varphi_x\right)$; $T_y = Z\left(\frac{\partial w}{\partial y} - \varphi_y\right)$

(ii) $T_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$; $T_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}$

$$M_x = -D\left(\frac{\partial \varphi_x}{\partial x} + \nu \frac{\partial \varphi_y}{\partial y}\right)$$

$$M_y = -D\left(\frac{\partial \varphi_y}{\partial y} + \nu \frac{\partial \varphi_x}{\partial x}\right)$$

$$M_{xy} = -D\,\frac{(1-\nu)}{2}\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right)$$

$$\frac{\partial T_x}{\partial y} - \frac{\partial T_y}{\partial x} = Z\left(\cancel{\frac{\partial^2 w}{\partial x \partial y}} - \frac{\partial \varphi_x}{\partial y} - \cancel{\frac{\partial^2 w}{\partial x \partial y}} + \frac{\partial \varphi_y}{\partial x}\right)$$

$$= -Z\left(\frac{\partial \varphi_x}{\partial y} - \frac{\partial \varphi_y}{\partial x}\right)$$

$$\frac{\partial T_x}{\partial y} - \frac{\partial T_y}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}\right)$$

$$= \boxed{\frac{\partial^2 M_x}{\partial x \partial y} + \frac{\partial^2 M_{xy}}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x^2} - \frac{\partial^2 M_y}{\partial x \partial y}}$$

$$\frac{\partial^2 M_x}{\partial x \partial y} = -D \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial \varphi_x}{\partial x} + \nu \frac{\partial \varphi_y}{\partial y}\right) = -D\left(\frac{\partial^3 \varphi_x}{\partial x^2 \partial y} + \nu \frac{\partial^3 \varphi_y}{\partial x \partial y^2}\right)$$

$$\frac{\partial^2 M_y}{\partial x \partial y} = -D \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial \varphi_y}{\partial y} + \nu \frac{\partial \varphi_x}{\partial x}\right) = -D\left(\frac{\partial^3 \varphi_y}{\partial x \partial y^2} + \nu \frac{\partial^3 \varphi_x}{\partial x^2 \partial y}\right)$$

$$\frac{\partial^2 M_{xy}}{\partial x^2} = -\frac{D(1-\nu)}{2} \frac{\partial^2}{\partial x^2}\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right) = -\frac{D(1-\nu)}{2}\left(\frac{\partial^3 \varphi_x}{\partial x^2 \partial y} + \frac{\partial^3 \varphi_y}{\partial x^3}\right)$$

$$\frac{\partial^2 M_{xy}}{\partial y^2} = -\frac{D(1-\nu)}{2} \frac{\partial^2}{\partial y^2}\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right) = -\frac{D(1-\nu)}{2}\left(\frac{\partial^3 \varphi_x}{\partial y^3} + \frac{\partial^3 \varphi_y}{\partial x \partial y^2}\right)$$

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la parte senza ν si scrive così

$$\frac{1}{2} \nabla^2 \left(\frac{\partial \varphi_x}{\partial y} - \frac{\partial \varphi_y}{\partial x}\right)$$

similmente si trova per la parte che moltiplica ν

$$\frac{\partial T_x}{\partial y} - \frac{\partial T_y}{\partial x} = -\frac{D(1-\nu)}{2} \nabla^2 \left(\frac{\partial \varphi_x}{\partial y} - \frac{\partial \varphi_y}{\partial x}\right)$$

$$-Z\left(\frac{\partial \varphi_x}{\partial y}-\frac{\partial \varphi_y}{\partial x}\right)=-\frac{D(1-\nu)}{2}\nabla^2\left(\frac{\partial \varphi_x}{\partial y}-\frac{\partial \varphi}{\partial x}\right)$$

$$\boxed{\nabla^2\left(\frac{\partial \varphi_x}{\partial y}-\frac{\partial \varphi_y}{\partial x}\right)=\frac{Z}{\frac{D(1-\nu)}{2}}\left(\frac{\partial \varphi_x}{\partial y}-\frac{\partial \varphi_y}{\partial x}\right)}$$