PIASTRA ALLA SOPHIE-GERMAIN – KIRCHHOFF

$$\left.\begin{aligned} t_x &= 0 \\ t_y &= 0 \end{aligned}\right\}$$ NO DEFORMABILITÀ AL TAGLIO

$$\Downarrow$$

CINEMATICA DEL MODELLO

$$\left.\begin{aligned} \varphi_x &= \frac{\partial w}{\partial x} \\ \varphi_y &= \frac{\partial w}{\partial y} \end{aligned}\right\} \Rightarrow \begin{aligned} S_x(x,y,z) &= u(x,y) - z\frac{\partial w}{\partial x} \\ S_y(x,y,z) &= v(x,y) - z\frac{\partial w}{\partial y} \\ S_z(x,y,z) &= w(x,y) \end{aligned}$$

IL PROBLEMA MEMBRANALE è analogo a quello che abbiamo studiato per la piastra R-M. Introduciamo il PROBLEMA FLESSIONALE

Deformazioni generalizzate $\Rightarrow$ curvature

$$\chi_x = -\frac{\partial \varphi_x}{\partial x} = -\frac{\partial^2 w}{\partial x^2}$$

$$\chi_y = -\frac{\partial \varphi_y}{\partial y} = -\frac{\partial^2 w}{\partial y^2}$$

$$\chi_{xy} = -\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right) = -2\frac{\partial^2 w}{\partial x \partial y}$$

Le quantità statiche coinvolte sono

$$M_x = \int_{-h/2}^{h/2} z\,\sigma_x\,dz \;;\quad M_y = \int_{-h/2}^{h/2} z\,\sigma_y\,dz \;;\quad M_{xy} = \int_{-h/2}^{h/2} z\,\tau_{xy}\,dz$$

$$p = \int_{-h/2}^{h/2} F_z\,dz$$

Adesso usiamo il principio dei lavori virtuali

$$\mathcal{L}_i = \int_A \left(M_x \, \delta\hat{\chi}_x + M_y \, \delta\hat{\chi}_y + M_{xy} \, \delta\hat{\chi}_{xy} \right) dA$$

$$= -\int_A \left(M_x \frac{\partial^2 \delta w}{\partial x^2} + M_y \frac{\partial^2 \delta w}{\partial y^2} + 2 M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y} \right) dA$$

$$\mathcal{L}_e = \int_A p \, \delta w \, dA + \int_\Gamma \left(V \delta w + W_n \underbrace{\frac{\partial \delta w}{\partial n}}_{\delta\varphi_n} + W_{sn} \underbrace{\frac{\partial \delta w}{\partial s}}_{\delta\varphi_s} \right) ds$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial n} \alpha_x - \frac{\partial w}{\partial s} \alpha_y$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial n} \alpha_y + \frac{\partial w}{\partial s} \alpha_x$$

Adesso vogliamo applicare il Teorema della divergenza e per farlo ci conviene riscrivere il $\mathcal{L}_i$

$$D[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$
$$f'(x)g(x) = D[f(x)g(x)] - f(x)g'(x)$$

$$M_x \frac{\partial^2 \delta w}{\partial x^2} = \frac{\partial}{\partial x}\left(M_x \frac{\partial \delta w}{\partial x}\right) - \frac{\partial M_x}{\partial x}\frac{\partial \delta w}{\partial x}$$

$$= \frac{\partial}{\partial x}\left(M_x \frac{\partial \delta w}{\partial x}\right) - \left[\frac{\partial}{\partial x}\left(\frac{\partial M_x}{\partial x}\delta w\right) - \delta w \frac{\partial^2 M_x}{\partial x^2}\right]$$

$$= \frac{\partial}{\partial x}\left(M_x \frac{\partial \delta w}{\partial x} - \frac{\partial M_x}{\partial x}\delta w\right) + \delta w \frac{\partial^2 M_x}{\partial x^2}$$

$$M_y \frac{\partial^2 \delta w}{\partial y^2} = \frac{\partial}{\partial y}\left(M_y \frac{\partial \delta w}{\partial y} - \frac{\partial M_y}{\partial y}\delta w\right) + \delta w \frac{\partial^2 M_y}{\partial y^2}$$

$$M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(M_{xy} \frac{\partial \delta w}{\partial y}\right) - \frac{\partial}{\partial y}\left(\delta w \frac{\partial M_{xy}}{\partial x}\right) + \delta w \frac{\partial^2 M_{xy}}{\partial x \partial y}$$

$$\mathcal{L}_i = -\int_A \delta w \left(\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2}\right) dA \; +$$

$$\text{A} \; + \int_A \left\{\frac{\partial}{\partial x}\left[\underbrace{\delta w \left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right)}_{g_x}\right] + \frac{\partial}{\partial y}\left[\underbrace{\delta w \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}\right)}_{g_y}\right]\right\} dA \; +$$

$$\text{B} \; + \int_A \left[\frac{\partial}{\partial x}\left(\underbrace{M_x \frac{\partial \delta w}{\partial x} + M_{xy}\frac{\partial \delta w}{\partial y}}_{h_x}\right) + \frac{\partial}{\partial y}\left(\underbrace{M_{xy}\frac{\partial \delta w}{\partial x} + M_y \frac{\partial \delta w}{\partial y}}_{h_y}\right)\right] dA$$

APPLICO IL TEOREMA DELLA DIVERGENZA A A e B

$$\mathcal{L}_i = -\int_A \delta w \left(\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} \right) dA +$$

$$\text{(A)} + \int_\Gamma \left[\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) \alpha_x + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} \right) \alpha_y \right] \delta w \, ds +$$

$$\text{(B)} + \int_\Gamma \left[\left(M_x \frac{\partial \delta w}{\partial x} + M_{xy} \frac{\partial \delta w}{\partial y} \right) \alpha_x + \left(M_{xy} \frac{\partial \delta w}{\partial x} + M_y \frac{\partial \delta w}{\partial y} \right) \alpha_y \right] ds$$

CONSIDERIAMO CHE $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial n}\frac{\partial n}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x}$

$$\frac{\partial n}{\partial x} = \alpha_x \qquad e \qquad \frac{\partial s}{\partial x} = -\alpha_y$$

$$\begin{cases} \frac{\partial w}{\partial x} = \frac{\partial w}{\partial n}\alpha_x - \frac{\partial w}{\partial s}\alpha_y \\ \frac{\partial w}{\partial y} = \frac{\partial w}{\partial n}\alpha_y + \frac{\partial w}{\partial s}\alpha_x \end{cases}$$

Quindi riscriviamo l'integrale (B)

$$\int_\Gamma \Big(M_x \frac{\partial \delta w}{\partial n}\alpha_x^2 - M_x \frac{\partial \delta w}{\partial s}\alpha_x\alpha_y + M_{xy}\frac{\partial \delta w}{\partial n}\alpha_y\alpha_x +$$

$$+ M_{xy}\frac{\partial \delta w}{\partial s}\alpha_x^2 + M_{xy}\frac{\partial \delta w}{\partial n}\alpha_x\alpha_y - M_{xy}\frac{\partial \delta w}{\partial s}\alpha_y^2 +$$

$$+ M_y \frac{\partial \delta w}{\partial n} \alpha_y^2 + M_y \frac{\partial \delta w}{\partial s} \alpha_x \alpha_y \Big) ds$$

derivate in direzione tangenziale

derivate in direzione normale

$$= \int_\Gamma \Big\{ \underbrace{\Big(M_x \alpha_x^2 + 2 M_{xy} \alpha_x \alpha_y + M_y \alpha_y^2 \Big)}_{M_n \text{ momento riportato al bordo}} \frac{\partial \delta w}{\partial n} +$$

$$+ \Big[\underbrace{(M_y - M_x) \alpha_x \alpha_y + M_{xy} (\alpha_x^2 - \alpha_y^2)}_{M_{sn} \text{ momento torcente riportato al bordo}} \Big] \frac{\partial \delta w}{\partial s} \Big\} ds$$

$$\mathcal{L}_i = - \int_A \delta w \left(\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} \right) dA +$$

$$+ \int_\Gamma T_n \, \delta w \, ds + \int_\Gamma \left(M_n \frac{\partial \delta w}{\partial n} + M_{sn} \frac{\partial \delta w}{\partial s} \right) ds$$

proiezione delle forze di taglio

$$T_n = \left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) \alpha_x + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} \right) \alpha_y$$

Invochiamo il Principio dei Lavori Virtuali

$$\mathcal{L}_i - \mathcal{L}_e = 0 \qquad \forall\, \delta w$$

$$-\int_A \left(\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\, \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p \right) \delta w \, dA \; +$$

$$+ \int_\Gamma \left[(T_n - V)\,\underline{\delta w} + (M_n + W_n)\frac{\partial \delta w}{\partial n} + \underline{(W_{sn} + M_{sn})\frac{\partial \delta w}{\partial s}} \right] ds = 0$$

su A

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\, \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p$$

oppure $\delta w = 0 \iff w = \bar{w}$

su Γ

$$M_n + W_n = 0 \qquad \text{oppure} \qquad \frac{\partial \delta w}{\partial n} = 0$$

$$\int_\Gamma (W_{sn} + M_{sn}) \frac{\partial \delta w}{\partial s}\, ds = \cancelto{0}{\int_\Gamma \frac{\partial}{\partial s}\left[(W_{sn} + M_{sn})\,\delta w \right] ds} \; +$$

perché il contorno è chiuso

$$- \int_\Gamma \left(\frac{\partial W_{sn}}{\partial s} + \frac{\partial M_{sn}}{\partial s} \right) \delta w \, ds$$

Questo dà luogo a

$$\int_\Gamma \left(V - T_n - \frac{\partial W_{sn}}{\partial s} - \frac{\partial M_{sn}}{\partial s} \right) \delta w \, ds = 0$$

che produce l'ultima condizione al contorno

$$T_n + \frac{\partial M_{sn}}{\partial s} = V - \frac{\partial W_{sn}}{\partial s} \quad \text{su } \Gamma$$

oppure $\quad w = \bar{w}$

taglio di Kirchhoff $\quad \bar{T}_n = \frac{\partial M_{sn}}{\partial s}$

EQUAZIONE DELLA PIASTRA DI KIRCHHOFF

Legame costitutivo

$$M_x = D(\chi_x + \nu \chi_y) \; ; \; M_y = D(\chi_y + \nu \chi_x)$$

$$M_{xy} = D \frac{(1-\nu)}{2} \chi_{xy}$$

$$D = \frac{E h^3}{12(1-\nu^2)}$$

Esplicitando le curvature

$$M_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)$$

$$M_y = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right)$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p$$

$$+D\left[\left(\frac{\partial^4 w}{\partial x^4} + \nu\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) - 2\cancel{D}(1-\nu)\frac{\partial^2 w}{\partial x^2 \partial y^2} - \cancel{D}\left(\frac{\partial^4 w}{\partial y^4} + \nu\frac{\partial^4 w}{\partial x^2 \partial y^2}\right)\right] = -p$$

$$\cancel{-\nu\frac{\partial^4 w}{\partial x^2 \partial y^2}} \qquad \cancel{+2\nu\frac{\partial^2 w}{\partial x^2 \partial y^2}} \qquad \cancel{-\nu\frac{\partial^4 w}{\partial x^2 \partial y^2}}$$

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}}$$

equazione della piastra di Sophie-Germain - Kirchhoff

$$\nabla_4(\cdot) = \frac{\partial^4(\cdot)}{\partial x^4} + 2\frac{\partial^4(\cdot)}{\partial x^2 \partial y^2} + \frac{\partial^4(\cdot)}{\partial y^4}$$

$$\nabla_4 w = \frac{p}{D} \qquad w(x,y)$$

$$T_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

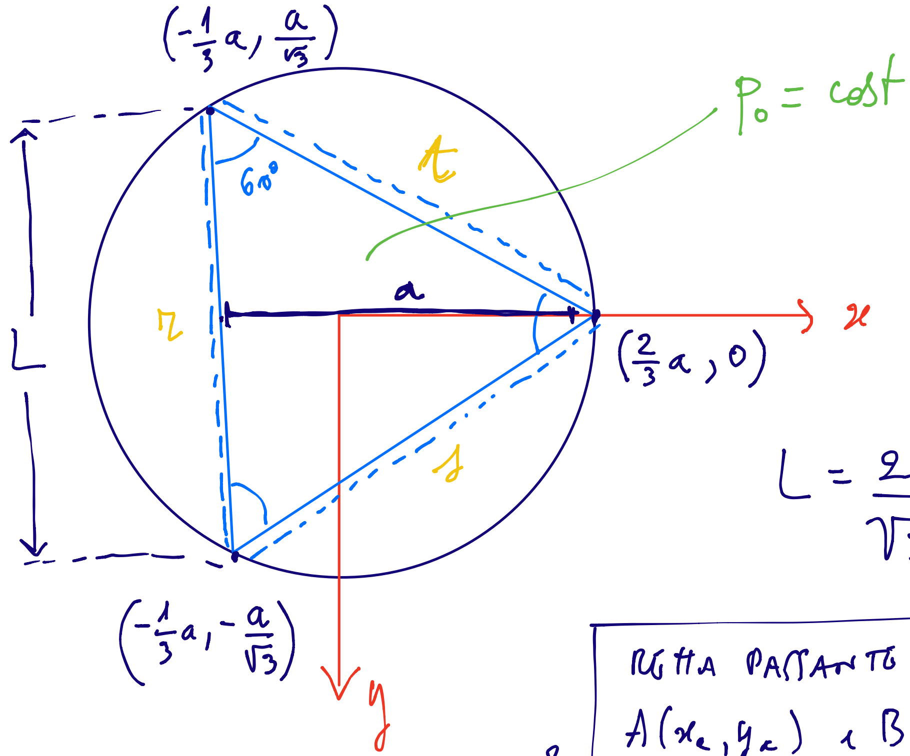
$$T_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}$$

$$T_x = -D\left(\frac{\partial^3 w}{\partial x^3} + \nu \frac{\partial^3 w}{\partial x \partial y^2}\right) - D(1-\nu)\frac{\partial^3 w}{\partial x \partial y^2}$$

$$= -D\frac{\partial^3 w}{\partial x^3} \cancel{- D\nu \frac{\partial^3 w}{\partial x \partial y^2}} - D\frac{\partial^3 w}{\partial x \partial y^2} \cancel{+ D\nu \frac{\partial^3 w}{\partial x \partial y^2}}$$

$$= -D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right)$$

ESEMPIO: PIASTRA A FORMA DI TRIANGOLO EQUILATERO CON BORDO APPOGGIATO E CARICO DISTRIBUITO COSTANTE



$$L = \frac{2a}{\sqrt{3}}$$

RETTA PASSANTE PER DUE PUNTI
$A(x_a, y_a)$ e $B(x_b, y_b)$

$$\frac{x - x_a}{x_b - x_a} = \frac{y - y_a}{y_b - y_a}$$

c) $x^2 + y^2 = \left(\frac{2}{3}a\right)^2$

r) $x = -\frac{1}{3}a$

s) $y\sqrt{3} = \frac{2}{3}a + x$

t) $y\sqrt{3} = \frac{2}{3}a - x$

$$W(x,y) = C_0 \underbrace{\left(x + \frac{1}{3}a\right)}_{r} \underbrace{\left(x - \frac{2}{3}a + y\sqrt{3}\right)}_{t} \underbrace{\left(x - \frac{2}{3}a - y\sqrt{3}\right)}_{s} \underbrace{\left(\frac{4}{9}a^2 - x^2 - y^2\right)}_{c}$$

$$= C_0\left(x^2 - \frac{\cancelto{1}{2}}{3}ax + xy\sqrt{3} + \cancel{\frac{1}{3}ax} - \frac{2}{9}a^2 + \frac{\sqrt{3}}{3}ay\right)\left(x - \frac{2}{3}a - y\sqrt{3}\right)\cdot$$

$$\cdot\left(\frac{4}{9}a^2 - x^2 - y^2\right)$$

$$= C_0\left(x^3 - \frac{2}{3}ax^2 - x^2y\sqrt{3} - \frac{1}{3}ax^2 + \frac{2}{9}a^2x + \frac{\sqrt{3}}{3}axy + \right.$$

$$+ x^2y\sqrt{3} - \frac{2\sqrt{3}}{3}axy - 3xy^2 - \frac{2}{9}a^2x + \frac{4}{27}a^3 + \frac{2\sqrt{3}}{9}a^2y +$$

$$\left. + \frac{\sqrt{3}}{3}axy - \frac{2\sqrt{3}}{9}a^2y - ay^2\right)\left(\frac{4}{9}a^2 - x^2 - y^2\right)$$