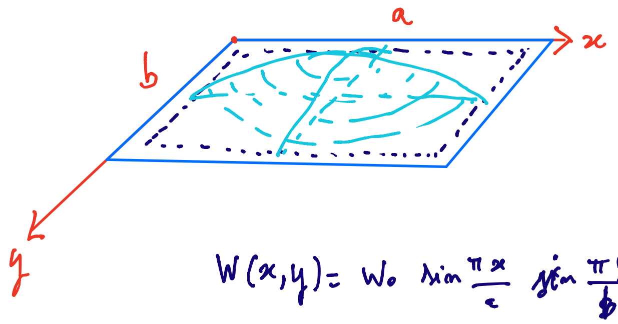
PIASTRA RETTANGOLARE CON BORDI APPOGGIATI E SOGGETTA A CARICO A BOLLA



$$p(x,y) = p_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$p\left(\frac{a}{2}, \frac{b}{2}\right) = p_0$$

$$W(x,y) = W_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

Consideriamo l'equazione della piastra $\nabla_4 W = \frac{p}{D}$

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \frac{p_0}{D} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

CONDIZIONI AL BORDO

LATI $x=0$ e $x=a$ $\longrightarrow$ $W=0$; $M_x=0$

LATI $y=0$ e $y=b$ $\longrightarrow$ $W=0$; $M_y=0$

Faccio le derivate (FARE LE DERIVATE!)

$$\frac{\partial W}{\partial x} = W_0 \frac{\pi}{a} \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$\boxed{\frac{\partial^2 W}{\partial x^2} = -W_0\left(\frac{\pi}{a}\right)^2 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}}$$

$$\frac{\partial^3 W}{\partial x^3} = -W_0\left(\frac{\pi}{a}\right)^3 \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$\frac{\partial^4 W}{\partial x^4} = W_0\left(\frac{\pi}{a}\right)^4 \underbrace{\sin\frac{\pi x}{a} \sin\frac{\pi y}{b}}$$

$$\frac{\partial W}{\partial y} = W_0 \frac{\pi}{b} \sin\frac{\pi x}{a} \cos\frac{\pi y}{b}$$

$$\frac{\partial^2 W}{\partial y^2} = -W_0\left(\frac{\pi}{b}\right)^2 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$\frac{\partial^3 W}{\partial y^3} = -W_0\left(\frac{\pi}{b}\right)^3 \sin\frac{\pi x}{a} \cos\frac{\pi y}{b}$$

$$\frac{\partial^4 W}{\partial y^4} = W_0\left(\frac{\pi}{b}\right)^4 \underbrace{\sin\frac{\pi x}{a} \sin\frac{\pi y}{b}}$$

$$\frac{\partial^3 w}{\partial x^2 \partial y} = - W_0 \left(\frac{\pi}{a}\right)^2 \left(\frac{\pi}{b}\right) \sin\frac{\pi x}{a} \cos\frac{\pi y}{b} \quad ; \quad \frac{\partial^4 W}{\partial x^2 \partial y^2} = W_0 \left(\frac{\pi^4}{a^2 b^2}\right) \underbrace{\sin\frac{\pi x}{a} \sin\frac{\pi y}{b}}$$

$$W_0 \left(\frac{\pi^4}{a^4} + 2 \frac{\pi^4}{a^2 b^2} + \frac{\pi^4}{b^4} \right) \cancel{\sin\frac{\pi x}{a}} \cancel{\sin\frac{\pi y}{b}} = \frac{P_0}{D} \cancel{\sin\frac{\pi x}{a}} \cancel{\sin\frac{\pi y}{b}}$$

$$W_0 = \frac{P_0}{D} \left(\frac{\pi^4 b^4 + 2\pi^4 a^2 b^2 + \pi^4 a^4}{a^4 b^4} \right)^{-1} =$$

$$= \boxed{\frac{P_0}{D} \frac{a^4 b^4}{\pi^4 (a^2+b^2)^2} = W_0}$$

$$W(x,y) = \frac{P_0 a^4 b^4}{\pi^4 D (a^2+b^2)^2} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

Verifico le condizioni al bordo

1. $W(x=0; y) = W(x=a; y) = W(x; y=0) = W(x; y=b) = 0$

è ovviamente verificata!

2. $M_x(x=0; y) = M_x(a; y) = M_y(x; 0) = M_y(x; b) = 0$

$$M_x = -D\left(\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2}\right) \quad ; \quad M_y = -D\left(\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2}\right)$$

ANCHE QUESTA E' OVVIAMENTE VERIFICATA!

$$W_{max} = W\left(\frac{a}{2}; \frac{b}{2}\right) = W_0 \qquad \text{spostamento massimo}$$

I momenti sono

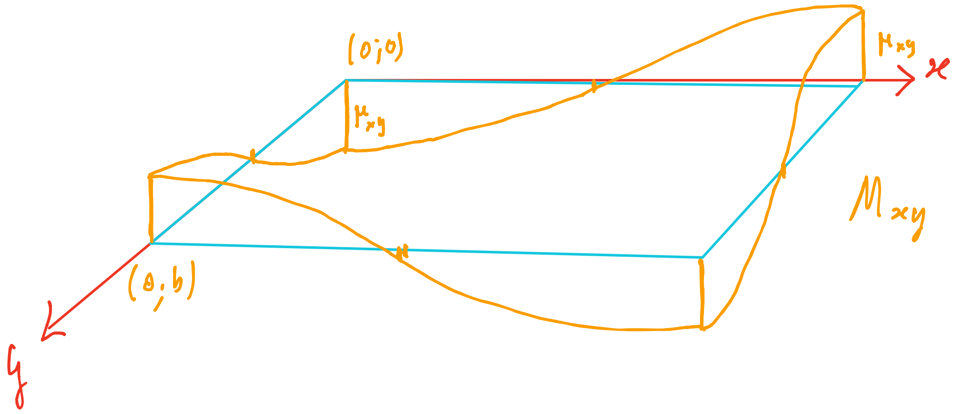
$$M_x = W_0 D \left(\frac{\pi^2}{a^2} + \nu \frac{\pi^2}{b^2}\right) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} = \mu_x \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$M_y = W_0 D \left(\frac{\pi^2}{b^2} + \nu \frac{\pi^2}{a^2}\right) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} = \mu_y \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

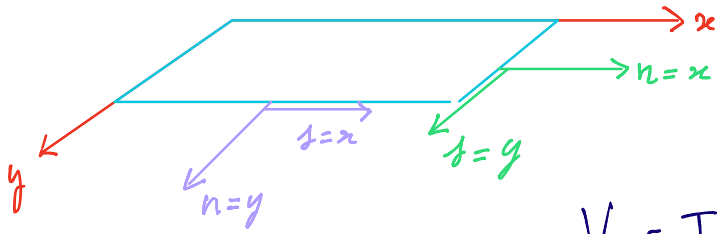
$$M_{xy} = -2D \frac{1-\nu}{2} \frac{\partial^2 W}{\partial x \partial y} = \mu_{xy} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b}$$

$$T_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = \rho_x \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$T_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = \rho_y \sin\frac{\pi x}{a} \cos\frac{\pi y}{b}$$



$$T_n + \frac{\partial M_{sn}}{\partial s} \qquad \text{taglio di Kirchhoff}$$



$$V_x = T_x + \frac{\partial M_{xy}}{\partial y}$$

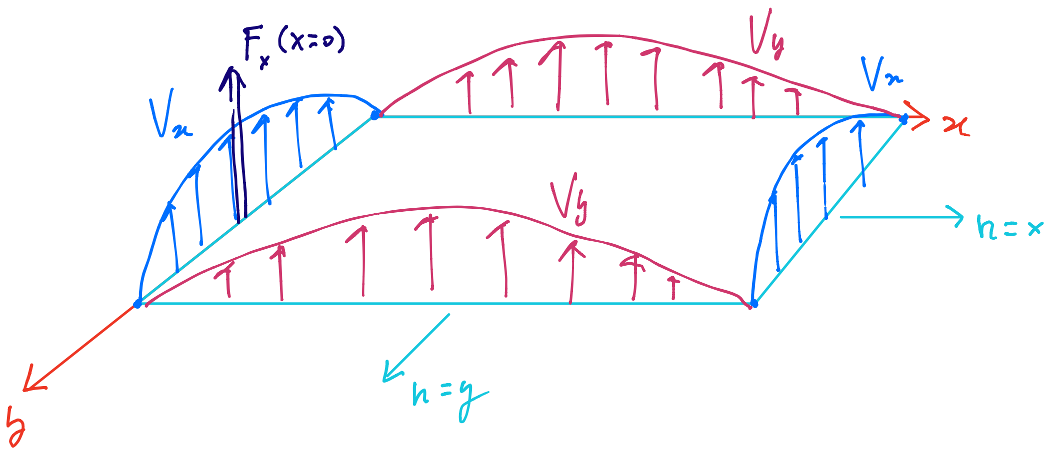
COMPITI A CASA:

FARE IN TOTO TUTTI I CONTI DI QUESTA LEZIONE

$$V_y = T_y + \frac{\partial M_{xy}}{\partial x}$$

$$V_x = \varphi_x \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$V_y = \varphi_y \sin\frac{\pi x}{a} \cos\frac{\pi y}{b}$$

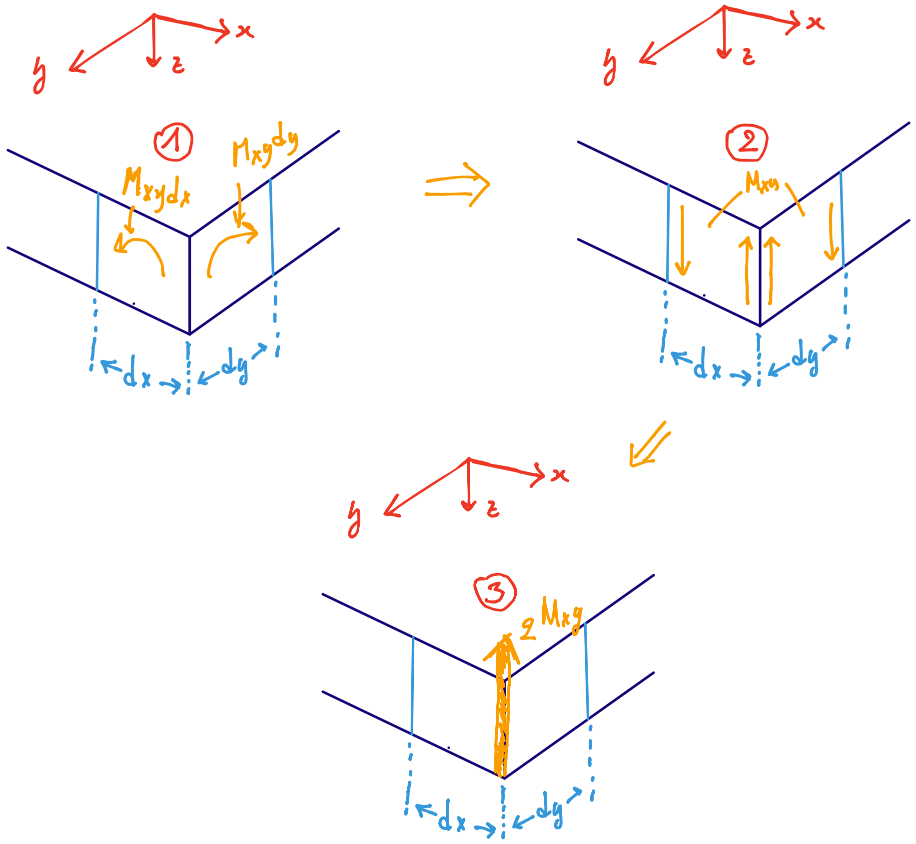


$$V_x = \varphi_x \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

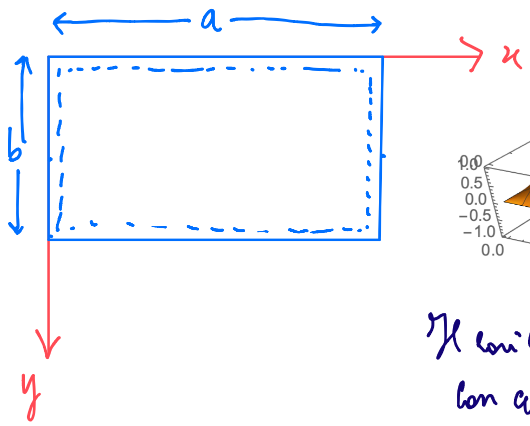
calcolo $V_x (x=0) = \varphi_x \sin\frac{\pi y}{b}$

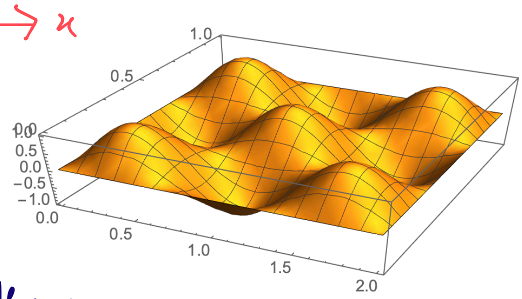
$$F_x(x=0) = \varphi_x \int_0^b \sin\frac{\pi y}{b} dy = \varphi_x \left[\frac{b}{\pi}\left(-\cos\frac{\pi y}{b}\right)\right]_0^b = \varphi_x \frac{2b}{\pi}$$

REAZIONI CONCENTRATE NEGLI SPIGOLI $R = -2M_{xy}$



SERIE DOPPIA DI NAVIER





Il carico è una funzione dei seni di x e y con certi coefficienti

$$p(x,y) = p_{nm} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$W(x,y) = W_{nm} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\nabla^4 W = \frac{p(x,y)}{D} \longrightarrow W_{nm} = \frac{p_{nm}}{\pi^4 D} \frac{1}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

se $n = m = 1$

$$W_{11} = \frac{p_{11}}{\pi^4 D} \frac{a^4 b^4}{(a^2+b^2)^2} \quad \left(= W_0 \right)$$

→ vedi esempio piastra rettangolare

$$\boxed{p(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}$$

$$p_{mn} = \frac{4}{ab} \int_0^a \int_0^b p(x,y) \sin\frac{m\pi x}{a} \sin\frac{m\pi y}{b} \, dx \, dy$$

$$\int_0^a \sin\frac{n\pi x}{a} \sin\frac{m\pi x}{a} \, dx = \begin{cases} 0 & \text{se } n \neq m \\ \frac{a}{2} & \text{se } n = m \end{cases}$$

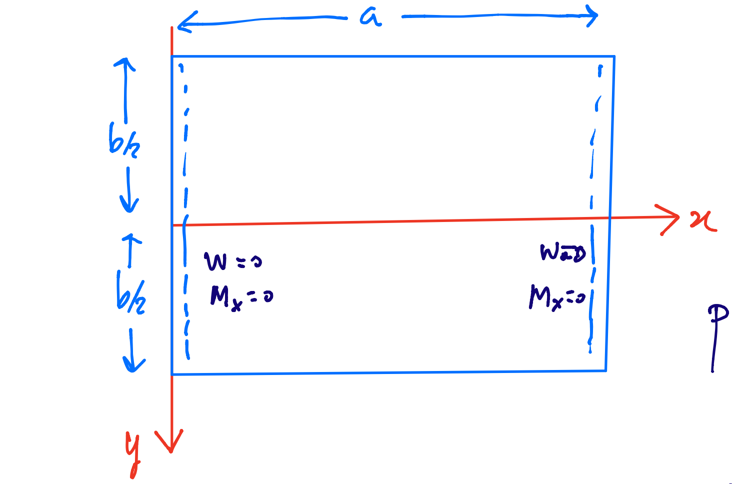
$$\int_0^b \sin\frac{q\pi y}{b} \sin\frac{n\pi y}{b} \, dy = \begin{cases} 0 & \text{se } q \neq n \\ \frac{b}{2} & \text{se } q = n \end{cases}$$

ORTOGONALITA' DEI COEFFICIENTI DI FOURIER

$$W(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$W_{nm} = \frac{p_{nm}}{\pi^4 D} \frac{1}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

SERIE SEMPLICE DI LEVY



Rappresentiamo il carico con uno sviluppo lungo x :

$$p(x,y) = \sum_{n=1}^{N} p_n(y) \sin \frac{n\pi x}{a}$$

$$\Downarrow$$

$$w(x,y) = \sum_{n=1}^{N} w_n(y) \sin \frac{n\pi x}{a}$$

$$\nabla^4 w = \frac{p}{D} \qquad \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}$$

FACCIO LE DERIVATE

- rispetto a x

per calcolare $\frac{\partial^4 w}{\partial x^2 \partial y^2}$

$$\frac{\partial w}{\partial x} = \frac{\pi}{a}\sum_{n=1}^{N} n\, w_n(y) \cos \frac{n\pi x}{a} \; ; \; \boxed{\frac{\partial^2 w}{\partial x^2} = -\left(\frac{\pi}{a}\right)^2 \sum_{n=1}^{N} n^2 w_n(y) \sin \frac{n\pi x}{a}}$$

$$\frac{\partial^3 w}{\partial x^3} = -\left(\frac{\pi}{a}\right)^3 \sum_{n=1}^{N} n^3 w_n(y) \cos \frac{n\pi x}{a} \; ; \; \frac{\partial^4 w}{\partial x^4} = \left(\frac{\pi}{a}\right)^4 \sum_{n=1}^{N} n^4 w_n(y) \sin \frac{n\pi x}{a}$$

- rispetto a y

$$\frac{\partial w}{\partial y} = \sum_{n=1}^{N} w'_n(y) \sin \frac{n\pi x}{a} \; ; \; \frac{\partial^2 w}{\partial y^2} = \sum_{n=1}^{N} w''_n(y) \sin \frac{n\pi x}{a}$$

$$\frac{\partial^3 w}{\partial y^3} = \sum_{n=1}^{N} w'''_n(y) \sin \frac{n\pi x}{a} \; ; \; \frac{\partial^4 w}{\partial y^4} = \sum_{n=1}^{N} w^{IV}_n(y) \sin \frac{n\pi x}{a}$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} = -\left(\frac{\pi}{a}\right)^2 \sum_{n=1}^{N} n^2 W_n''(y) \sin\frac{n\pi x}{a}$$

$$\sum_{n=1}^{N}\left[n^4\left(\frac{\pi}{a}\right)^4 W_n(y) - 2n^2\left(\frac{\pi}{a}\right)^2 W_n''(y) + W_n^{IV}(y)\right]\sin\frac{n\pi x}{a} = \sum_{n=1}^{N}\frac{P_n(y)}{D}\sin\frac{n\pi x}{a}$$

$$\Downarrow$$

$$n^4\left(\frac{\pi}{a}\right)^4 W_n(y) - 2n^2\left(\frac{\pi}{a}\right)^2 W_n''(y) + W_n^{IV}(y) = \frac{P_n(y)}{D}$$

$$\boxed{W_n^{IV}(y) - 2\left(\frac{n\pi}{a}\right)^2 W_n''(y) + \left(\frac{n\pi}{a}\right)^4 W_n(y) = \frac{P_n(y)}{D}}$$

equazione differenziale nell'incognita

$W_1(y), W_2(y), \dots, W_N(y)$

$$W_n(y) = W_{n,om}(y) + W_{n,part}(y)$$

Consideriamo l'equazione omogenea

$$W_n^{IV}(y) - 2\left(\frac{n\pi}{a}\right)^2 W_n''(y) + \left(\frac{n\pi}{a}\right)^4 W_n(y) = 0$$

$$W_n(y) = C e^{\lambda y}$$

$$W_n'(y) = C\lambda e^{\lambda y} \; ; \; W_n''(y) = C\lambda^2 e^{\lambda y} \; ; \; W_n'''(y) = C\lambda^3 e^{\lambda y}$$

$$W_n^{IV}(y) = C\lambda^4 e^{\lambda y}$$

$$C e^{\lambda y}\underbrace{\left[\lambda^4 - 2\left(\frac{n\pi}{a}\right)^2\lambda^2 + \left(\frac{n\pi}{a}\right)^4\right]}_{\text{equazione caratteristica}} = 0$$

poniamo $\lambda^2 = t$

$$t^2 - 2\left(\frac{n\pi}{a}\right)^2 t + \left(\frac{n\pi}{a}\right)^4 = 0$$

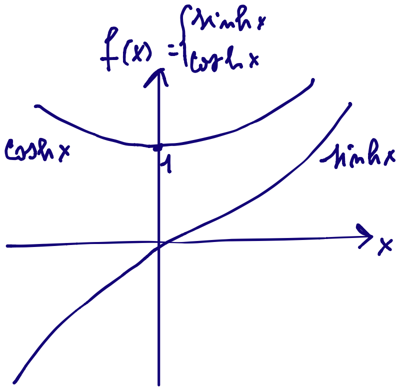
$$t_{1,2} = \left(\frac{n\pi}{a}\right)^2 \Rightarrow \begin{cases} \lambda_1 = \frac{n\pi}{a} \\ \lambda_2 = \frac{n\pi}{a} \\ \lambda_3 = -\frac{n\pi}{a} \\ \lambda_4 = -\frac{n\pi}{a} \end{cases}$$

Quindi la soluzione dell'omogenea è

$$W_n(y) = A'_n e^{\frac{n\pi}{a}y} + B'_n \frac{n\pi}{a} y e^{\frac{n\pi}{a}y} + C'_n e^{-\frac{n\pi}{a}y} + D'_n \left(-\frac{n\pi y}{a}\right) e^{-\frac{n\pi}{a}y}$$

$$\frac{e^x + e^{-x}}{2} = \cosh x \quad ; \quad \frac{e^x - e^{-x}}{2} = \sinh x$$

$$W_n(y) = A_n \cosh \underbrace{\frac{n\pi y}{a}}_{\text{pari}} + B_n \underbrace{\frac{n\pi y}{a}}_{\text{disp}} \cdot \underbrace{\sinh \frac{n\pi y}{a}}_{\text{disp} = \text{pari}} + C_n \underbrace{\sinh \frac{n\pi y}{a}}_{\text{disp}} + D_n \underbrace{\frac{n\pi y}{a}}_{\text{disp}} \cdot \underbrace{\cosh \frac{n\pi y}{a}}_{\text{pari} = \text{disp}}$$



SE x è asse di simmetria per la piastra (simmetrica rispetto ad y) allora alcuni pesi devono essere nulli

$$\Downarrow$$

$$C_n = D_n = 0$$

La soluzione completa è

$$W_n(y) = A_n \cosh \frac{n\pi y}{a} + B_n \frac{n\pi y}{a} \sinh \frac{n\pi y}{a} + \left(\frac{a}{n\pi}\right)^4 \frac{p_n}{D}$$