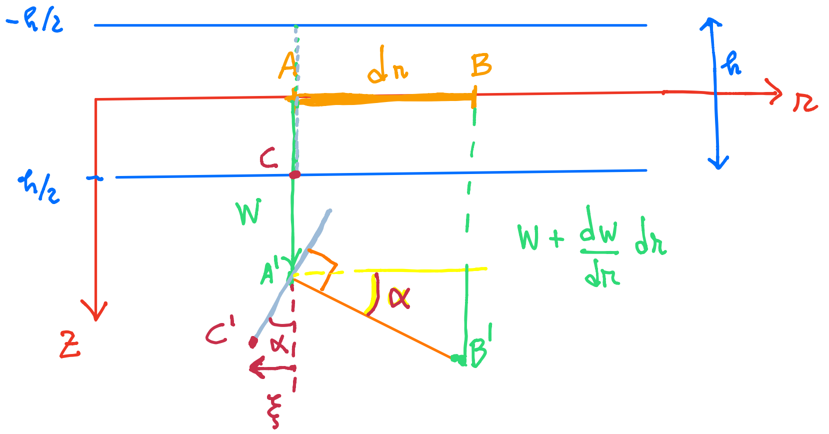
STIAMO PARLANDO DI Piastre circolari assialsimmetriche (z, r, ϑ)

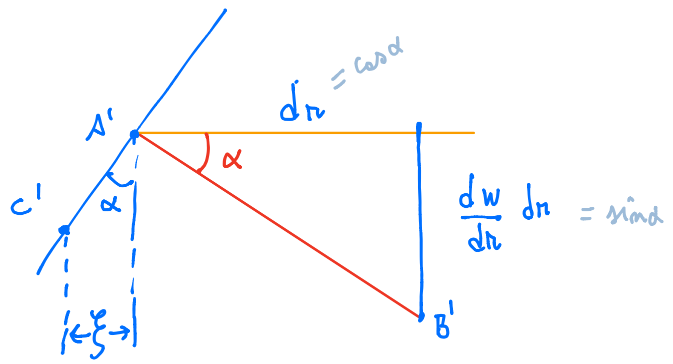
i. $$\frac{d}{dr}(r\,T_r) = p\,r$$

ii. $$M_r - M_\vartheta + r\frac{dM_r}{dr} + r\,T_r = 0$$

iii. $$\frac{d}{dr}(M_r - M_\vartheta) + \frac{d}{dr}\left(r\frac{dM_r}{dr}\right) + p\,r = 0$$

CINEMATICA



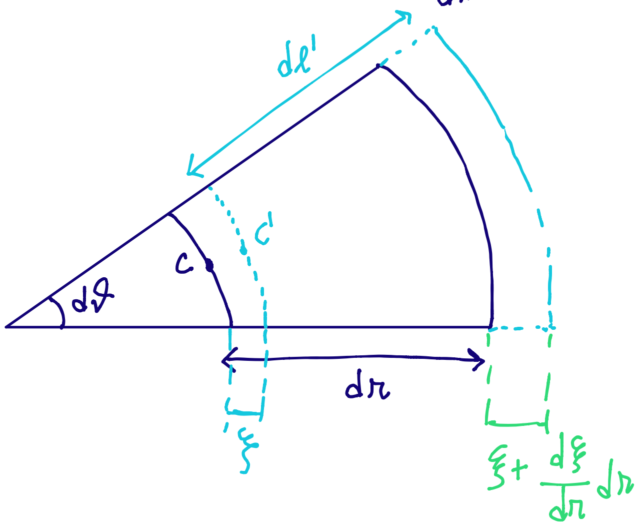


NOTA! siamo in piccole deformazioni $\Rightarrow \operatorname{tg}\alpha \approx \alpha$

$$\operatorname{tg}\alpha = \frac{\frac{dw}{dr}\cancel{dr}}{\cancel{dr}} = \frac{dw}{dr}$$

$$\alpha \simeq \frac{dw}{dr} \quad ; \quad \xi = z \sin\alpha \simeq z\alpha = z\frac{dw}{dr}$$

$$\xi = -z\frac{dw}{dr}$$



$$\varepsilon_r = \frac{dl' - dr}{dr}$$

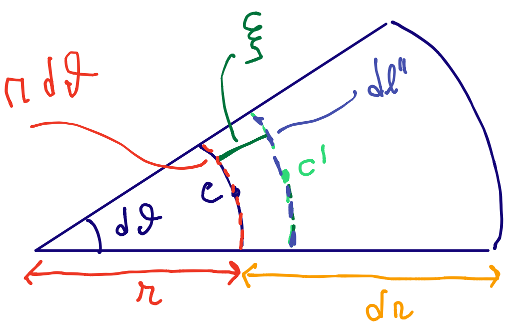
$$dl' = dr - \xi + \left(\xi + \frac{d\xi}{dr}dr\right)$$

$$= dr + \frac{d\xi}{dr}dr$$

$$\varepsilon_r = \frac{dl' - dr}{dr} = \frac{\cancel{dr} + \frac{d\xi}{dr}dr - \cancel{dr}}{dr} = \frac{d\xi}{dr}$$

$$\boxed{\varepsilon_r = \frac{d\xi}{dr}} = \frac{d}{dr}\left(-z\frac{dw}{dr}\right) = -z\frac{d^2w}{dr^2}$$

deformazione radiale



$$\varepsilon_\vartheta = \frac{dl'' - r\,d\vartheta}{r\,d\vartheta}$$

$$= \frac{(\cancel{r} + \xi)\cancel{d\vartheta} - \cancel{r}\cancel{d\vartheta}}{r\cancel{d\vartheta}}$$

$$= \frac{\xi}{r} = -\frac{z}{r}\frac{dw}{dr}$$

$$\boxed{\varepsilon_\vartheta = -\frac{z}{r}\frac{dw}{dr}}$$

deformazione circonferenziale

LEGAME COSTITUTIVO

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r - \nu\,\varepsilon_\vartheta\right) = -\frac{Ez}{1-\nu^2}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$$

$$\sigma_\vartheta = \frac{E}{1-\nu^2}\left(\varepsilon_\vartheta - \nu\,\varepsilon_r\right) = -\frac{Ez}{1-\nu^2}\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right)$$

$$M_r = \int_{-h/2}^{h/2} \sigma_r\, z\, dz = -\frac{E}{1-\nu^2}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)\int_{-h/2}^{h/2} z^2\, dz$$

$$= -\overbrace{\frac{E h^3}{12(1-\nu^2)}}^{D}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$$

$$M_r = -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$$

$$M_\vartheta = -D\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right)$$

ii. $M_r - M_\vartheta + r\dfrac{dM_r}{dr} + r\,T_r = 0$

$$T_r = \frac{1}{r}\left(M_\theta - M_r - r\frac{dM_r}{dr}\right)$$

$$= \frac{D}{r}\left\{\left[-\frac{1}{r}\frac{dw}{dr} - \nu\frac{d^2w}{dr^2} + \frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right] + r\left[\frac{d^3w}{dr^3} + \nu\left(-\frac{1}{r^2}\frac{dw}{dr} + \frac{1}{r}\frac{d^2w}{dr^2}\right)\right]\right\}$$

$$= \frac{D}{r}\left(-\frac{1}{r}\frac{dw}{dr} \cancel{-\nu\frac{d^2w}{dr^2}} + \frac{d^2w}{dr^2} \cancel{+\frac{\nu}{r}\frac{dw}{dr}} + r\frac{d^3w}{dr^3} \cancel{-\frac{\nu}{r}\frac{dw}{dr}} \cancel{+\nu\frac{d^2w}{dr^2}}\right)$$

$$= D\left(\frac{d^3w}{dr^3} + \underbrace{\frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}}_{\frac{d}{dr}\left(\frac{1}{r}\frac{dw}{dr}\right)}\right)$$

$$T_r = D\left[\frac{d^3w}{dr^3} + \frac{d}{dr}\left(\frac{1}{r}\frac{dw}{dr}\right)\right]$$

L'equazione d'equilibrio era

$$\frac{d}{dr}\left(M_r - M_\theta + r\frac{dM_r}{dr}\right) + pr = 0$$

$$\frac{d}{dr}\left(M_\theta - M_r - r\frac{dM_r}{dr}\right) = pr$$

$$\frac{d}{dr}\left(-\frac{D}{r}\frac{dw}{dr} - \cancel{D\nu\frac{d^2w}{dr^2}} + D\frac{d^2w}{dr^2} + \cancel{\frac{D\nu}{r}\frac{dw}{dr}} + Dr\frac{d^3w}{dr^3} - \cancel{\frac{rD\nu}{r^2}\frac{dw}{dr}} + \cancel{\frac{rD\nu}{r}\frac{d^2w}{dr^2}}\right) = pr$$

$$\boxed{\frac{d}{dr}\left(r\frac{d^3w}{dr^3} + \frac{d^2w}{dr^2} - \frac{1}{r}\frac{dw}{dr}\right) = \frac{pr}{D}}$$

equazione delle piastre

$$\frac{d^3w}{dr^3} + r\frac{d^4w}{dr^4} + \frac{d^3w}{dr^3} + \frac{1}{r^2}\frac{dw}{dr} - \frac{1}{r}\frac{d^2w}{dr^2} = \frac{pr}{D}$$

$$\frac{d^4w}{dr^4} + \frac{2}{r}\frac{d^3w}{dr^3} - \frac{1}{r^2}\frac{d^2w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr} = \frac{p}{D}$$

$$r\frac{d^3w}{dr^3} + \frac{d^2w}{dr^2} - \frac{1}{r}\frac{dw}{dr} = \int \frac{pr}{D}\,dr + C_1$$

dividendo tutto per r

$$\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} = \frac{1}{r}\int \frac{pr}{D}\,dr + \frac{C_1}{r}$$

$$\frac{d}{dr}\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = \frac{1}{r}\int \frac{pr}{D}\,dr + \frac{C_1}{r}$$

integro ancora

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} = \int \frac{1}{r}\,dr \int \frac{pr}{D}\,dr + C_1 \ln r + C_2$$

moltiplico tutto per r

$$r\frac{d^2w}{dr^2} + \frac{dw}{dr} = r\int \frac{1}{r}\,dr \int \frac{pr}{D}\,dr + C_1 r\ln r + C_2 r$$

$$\frac{d}{dr}\left(r\frac{dw}{dr}\right) = r\int \frac{1}{r}\,dr \int \frac{pr}{D}\,dr + C_1 \underbrace{r\ln r}_{\frac{d}{dr}\left(\frac{r^2}{2}\ln r - \frac{r^2}{4}\right)} + C_2 r$$

integro per la III volta

$$r\frac{dw}{dr} = \int r\,dr \int \frac{1}{r}\,dr \int \frac{pr}{D}\,dr + C_1\left(\frac{r^2}{2}\ln r - \frac{r^2}{4}\right) + C_2\frac{r^2}{2} + C_3$$

$$\frac{d}{dx}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right) = \frac{2x}{2}\ln x + \frac{x^2}{2}\frac{1}{\cancel{x}} - \frac{2x}{4}$$

$$= x\ln x + \frac{x}{2} - \frac{x}{2} = x\ln x$$

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divido ancora un'ultima volta per r

$$\frac{dw}{dr} = \frac{1}{r}\int r\,dr \int \frac{1}{r}\,dr \int \frac{pr}{D}\,dr + C_1\left(\frac{r}{2}\ln r - \frac{r}{4}\right) + C_2\frac{r}{2} + \frac{C_3}{r}$$

integro per l'ultima volta

$$w(r) = \int \frac{1}{r}\,dr \int r\,dr \int \frac{1}{r}\,dr \int \frac{pr}{D}\,dr + C_1\left[\frac{1}{2}\left(\frac{r^2}{2}\ln r - \frac{r^2}{4}\right) - \frac{r^2}{8}\right] + C_2\frac{r^2}{4} + C_3 \ln r + C_4$$

Ridefinisco le costanti. (FATE IL CONTO!)

$$A_1 = \frac{C_1}{4} \; ; \; A_2 = \frac{C_2 - C_1}{4} \; ; \; A_3 = C_3 \; ; \; A_4 = C_4$$

$$w(r) = \underbrace{A_1 r^2 \ln r + A_2 r^2 + A_3 \ln r + A_4}_{w_1} + \underbrace{\int \frac{1}{r}\,dr \int r\,dr \int \frac{1}{r}\,dr \int \frac{pr}{D}\,dr}_{w_0}$$

Per piastre "piene" (senza foro e quindi doppio bordo)

1) lo spostamento al centro deve essere finito

2) le azioni interne devono essere finite

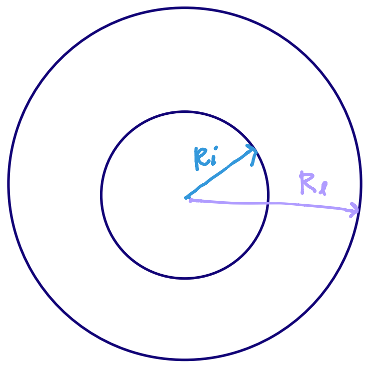
1. $\lim_{r \to 0} w(r)$ deve essere finito $\Rightarrow A_3 = 0$

2. $\left.\begin{array}{l} \dfrac{dw}{dr} \text{ e } \dfrac{d^2 w}{dr^2}, \quad \lim\limits_{r \to 0} \dfrac{dw}{dr} \\ \qquad\qquad\qquad\;\; \lim\limits_{r \to 0} \dfrac{d^2 w}{dr^2} \end{array}\right\}$ siano finiti

$$\left.\begin{array}{l} \dfrac{dw_1}{dr} = A_1\left(2r \ln r + r^2 \dfrac{1}{r}\right) + 2A_2 r + A_3 \dfrac{1}{r} \\[2ex] \dfrac{d^2 w_1}{dr^2} = A_1\left(2 \ln r + 2r\dfrac{1}{r} + 1\right) + 2A_2 - \dfrac{A_3}{r^2} \end{array}\right\} \Rightarrow A_1 = 0$$

CONDIZIONI AL BORDO

(indifferentemente per R_e e R_i)



1) bordo incastrato

$$W(r=R)=0$$ abbassamento nullo

$$\left.\frac{dw}{dr}\right|_{r=R}=0$$ rotazione impedita

2) Bordo appoggiato

$$W(r=R)=0$$ abbassamento nullo

$$M_r(r=R)=0$$ momento nullo

$$\Downarrow$$

$$\left.-D\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right)\right|_{r=R}=0$$

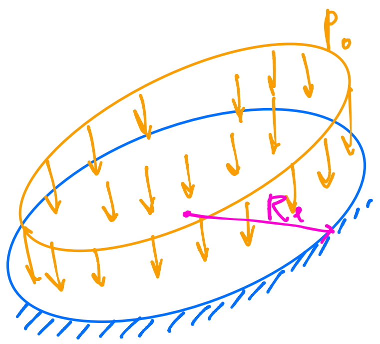
3) Bordo libero

$$M_r(r=R)=0 \Rightarrow \left.-D\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right)\right|_{r=R}=0$$

$$T_r(r=R)=0$$

$$\left.D\left(\frac{d^3w}{dr^3}+\frac{1}{r}\frac{d^2w}{dr^2}-\frac{1}{r^2}\frac{dw}{dr}\right)\right|_{r=R}=0$$

PIASTRA CON BORDO INCASTRATO E CARICO UNIFORME



$$w_0 = \int \frac{1}{r}\,dr \int r\,dr \int \frac{1}{r}\,dr \int \frac{p_0 r}{D}\,dr$$

$$W_0 = \frac{p_0 r^4}{64D}$$

$$W(r) = \frac{p_0 r^4}{64D} + \underbrace{\cancel{A_1}}_{0} r^2 \ln r + A_2 r^2 + \underbrace{\cancel{A_3}}_{0} \ln r + A_4$$

perchè la piastra è piena e non posso avere abbassamento o azioni interne che divergono

INCASTRO

$$\begin{cases} W(r = R_e) = 0 \\ \left.\dfrac{dW}{dr}\right|_{r=R_e} = 0 \end{cases}$$

$$W(r) = \frac{p_0 r^4}{64D} + A_2 r^2 + A_4$$

① $W(r = R_e) = 0$

$$\frac{p_0 R_e^4}{64D} + A_2 R_e^2 + A_4 = 0$$

$$A_4 = -\frac{p_0 R_e^4}{64D} - A_2 R_e^2$$

② $\left.\dfrac{dw}{dr}\right|_{r=R_e} = 0$

$$\frac{dw}{dr} = \frac{p_0 r^3}{16D} + 2A_2 r$$

$$\frac{p_0 R_e^3}{16D} + 2A_2 R_e = 0$$

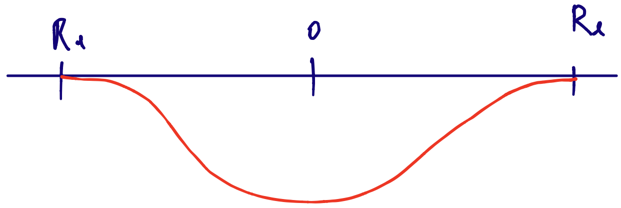
$$A_2 = -\frac{p_0 R_e^2}{32D}$$

$$A_4 = -\frac{p_0 R_e^4}{64D} + \frac{p_0 R_e^4}{32D} = \frac{p_0 R_e^4}{64D}$$

QUINDI

$$w(r) = \frac{p_0 r^4}{64D} - \frac{p_0 R_e^2}{32D} r^2 + \frac{p_0 R_e^4}{64D} = \frac{p_0}{64D}\left(R_e^2 - r^2\right)^2$$

$$W_{max} = W(0) = \frac{p_0 R_e^4}{64D}$$



CALCOLARE M_r, M_θ e T_r

Risultante del carico $R_p = \int p_0 \, dA = \int_0^{R_e} p_0 r \, dr \int_0^{2\pi} d\theta$

Risultante del Taglio $R_T = \oint T_r \, dA = \int_0^{2\pi} R_e T_r \, d\theta$