BASICS OF MICROPOLAR MECHANICS

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Università degli Studi di Cagliari, Cagliari, June, 2017

Contents

Some Historical Remarks

Original works

- W. Voigt (1887)
- E. & F. Cosserat (1909)
- J. L. Ericksen and C. Truesdell (1958)
- W. Günther (1958)
- G. Grioli (1960)
- R.D. Mindlin and H.F. Tiersten (1962), R. A. Toupin (1962)
- H. Schaefer (1962)
- E. V. Kuvshinskii and E.L. Aero (1963)
- A.C. Eringen and E.S. Suhubi (1964)
- V.A. Palmov (1964), W. T. Koiter (1964)
- ...

Some Historical Remarks. Cont'd

CISM Courses

- R. Stojanović. Mechanics of Polar Continua. Theory and Applications. Udine, 1969. No. 2.
- W. Nowacki. Theory of Micropolar Elasticity. Udine, 1970. No. 25.
- R. Stojanović. Recent Developments in the Theory of Polar Continua. Udine 1970. No. 27.
- W. Nowacki and W. Olszak. Micropolar Elasticity. Udine, 1974. No. 151.

Some Historical Remarks. Cont'd

Classical books

- Nowacki, W.: Theory of Asymmetric Elasticity. Pergamon-Press, Oxford et al. (1986)
- Eringen, A.C.: Microcontinuum Field Theory. I. Foundations and Solids. Springer, New York (1999)
- Eringen, A.C.: Microcontinuum Field Theory. II. Fluent Media. Springer, New York (2001)

Proceedings

- G.A. Maugin, A.V. Metrikine (eds.). Mechanics of Generalized Continua: One hundred years after the Cosserats. Springer, New York (2010)
- H. Altenbach, V.I. Erofeev, G.A. Maugin (eds.). Mechanics of Generalized Continua. From the Micromechanical Basics to Engineering Applications. Springer, Berlin (2011)

Motivation

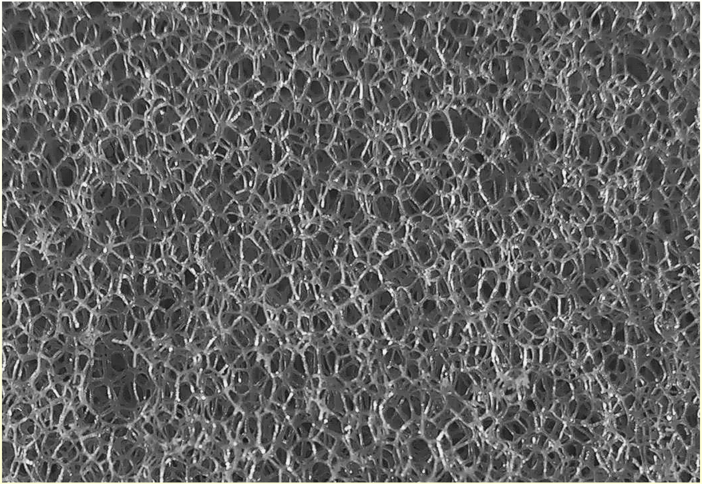


Figure: Polymer open-cell foam is the example of a complex medium modelled as a micropolar continuum

Notations

We use the following notations.

1. Vectors and tensors are denoted by semibold roman font like $\mathbf{A}$, $\mathbf{a}$.

2. Higher-order tensors are denoted by blackboard bold font like $\mathbb{A}$.

3. Greece indices take values 1 and 2, Latin indices are arbitrary.

4. Grad operator.

$$\begin{gathered}
[\mathrm{Grad}\mathbf{v}(\mathbf{x})]\cdot\mathbf{a} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(\mathbf{x}+t\mathbf{a})|_{t=0}\ , \\
[\mathrm{Grad}\mathbf{A}(\mathbf{x})]\cdot\mathbf{a} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(\mathbf{x}+t\mathbf{a})|_{t=0}\ , \quad \text{for any } t\in R\ ,\ \mathbf{a}\in E\ .
\end{gathered} \tag{1}$$

5. Div operator.

$$[\mathrm{Div}\mathbf{A}(\mathbf{x})]\cdot\mathbf{a} = \mathrm{Div}[\mathbf{A}(\mathbf{x})\cdot\mathbf{a}], \quad \forall\ \mathbf{A}\in E\otimes E, \quad \forall\ \mathbf{a}\in E. \tag{2}$$

Introduction: Mass-points Motion

Newton's law. Motion of one and n mass-points is determined by

$$m\dot{\mathbf{v}} = \mathbf{f}, \quad m_i\dot{\mathbf{v}}_i = \mathbf{f}_i, \qquad i = 1, 2, \dots, n. \tag{3}$$

Definition

The momentum of the mass-point called also the linear momentum, is the quantity $\mathfrak{P} = m\mathbf{v}$*. The moment of momentum of the mass-point with respect to a point with radius vector* $\mathbf{r}_0$ *called the pole is* $\mathfrak{M} = (\mathbf{r} - \mathbf{r}_0) \times m\mathbf{v}$*.*

Definition

The momentum and the moment of momentum of n *mass-points with respect pole* $\mathbf{r}_0$ *are*

$$\mathfrak{P} = \sum_{i=1}^{n} m_i\mathbf{v}_i, \quad \textit{and} \quad \mathfrak{M} = \sum_{i=1}^{n} (\mathbf{r}_i - \mathbf{r}_0) \times m_i\mathbf{v}_i.$$

Introduction: Mass-points Motion. Cont'd

Theorem

The rate of the change of the momentum of an n mass-points is equal to the total (resulting) force vector $\mathfrak{F}$, that is the sum of all the forces acting to the mass-points

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathfrak{P} = \mathfrak{F}, \qquad \mathfrak{F} \triangleq \sum_{i=1}^{n} \mathbf{f}_i. \tag{4}$$

Theorem

The rate of the change of the moment of momentum with respect to pole $\mathbf{r}_0$ of an n mass-points is equal to the total torque (resulting moment) $\mathfrak{C}$ with respect to pole $\mathbf{r}_0$ of all the forces acting on the mass-points

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathfrak{M} = \mathfrak{C}, \qquad \mathfrak{C} \triangleq \sum_{i=1}^{n} (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{f}_i. \tag{5}$$

Introduction: Mass-points Motion. Cont'd

Theorem

The rate of the change of the momentum of an n mass-points is equal to the total (resulting) force vector $\mathfrak{F}$, that is the sum of all the forces acting to the mass-points

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathfrak{P} = \mathfrak{F}, \qquad \mathfrak{F} \triangleq \sum_{i=1}^{n} \mathbf{f}_i. \tag{6}$$

Theorem

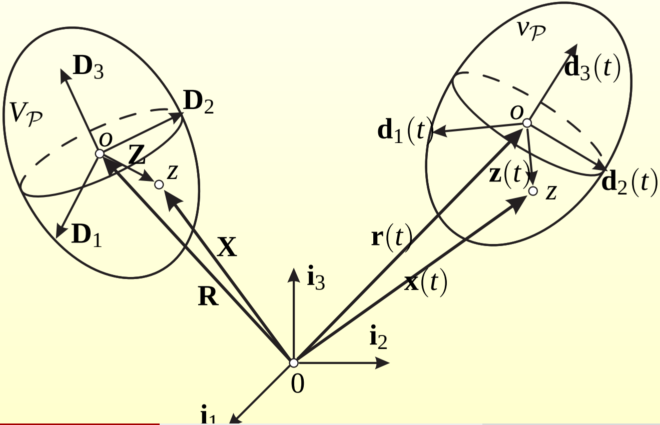
The rate of the change of the moment of momentum with respect to pole $\mathbf{r}_0$ of an n mass-points is equal to the total torque (resulting moment) $\mathfrak{C}$ with respect to pole $\mathbf{r}_0$ of all the forces acting on the mass-points

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathfrak{M} = \mathfrak{C}, \qquad \mathfrak{C} \triangleq \sum_{i=1}^{n} (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{f}_i. \tag{7}$$

Introduction: Rigid Body Kinematics

Definition

A set of material points for which the mutual distances between the points remain unchanged in motion, is called a rigid body.



Introduction: Rigid Body Kinematics. Cont'd

The position any point $z \in \mathcal{P}$ is given by

$$\mathbf{r}(t) = \mathbf{R}_0 + \mathbf{u}(t) + \mathbf{Q}(t) \cdot \mathbf{Z}, \tag{8}$$

where $\mathbf{Q} = \mathbf{d}_i \otimes \mathbf{D}_i$ is the rotation tensor.
Differentiating (8) we get

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{u}}(t) + \dot{\mathbf{Q}}(t) \cdot \mathbf{Z}. \tag{9}$$

$\mathbf{Q}$ is orthogonal so tensor $\dot{\mathbf{Q}} \cdot \mathbf{Q}^T$ is antisymmetric and it can be represented in the form

$$\dot{\mathbf{Q}} \cdot \mathbf{Q}^T = \boldsymbol{\omega} \times \mathbf{I}, \tag{10}$$

where $\boldsymbol{\omega}$ is the angular velocity of $\mathcal{P}$. Vector $\boldsymbol{\omega}$ can be determined as follows

$$\boldsymbol{\omega} = -\frac{1}{2}(\dot{\mathbf{Q}} \cdot \mathbf{Q}^T)_\times \tag{11}$$

Thus the velocity vector of a body point takes the form

$$\mathbf{v}(t) = \dot{\mathbf{u}}(t) + \boldsymbol{\omega}(t) \times \mathbf{Z}. \tag{12}$$

Introduction: Rigid Body Motion

The rigid body can be considered as a system of mass-points and so we can introduce the following definitions.

Definition

The momentum and the moment of momentum with respect to pole $\mathbf{r}_0$ for a rigid body are the quantities

$$\mathfrak{P} = \iiint\limits_{v_{\mathcal{P}}} \rho \mathbf{v}\, dv, \qquad \mathfrak{M} = \iiint\limits_{v_{\mathcal{P}}} \rho(\mathbf{r} - \mathbf{r}_0) \times \mathbf{v}\, dv,$$

respectively.

Here ρ is the mass density of $\mathcal{P}$ so its mass m is given by the integral over domain $v_{\mathcal{P}} \subset \mathbb{R}^3$ taken by the body in space, $m(\mathcal{P}) = \iiint\limits_{v_{\mathcal{P}}} \rho\, dv$.

Introduction: Rigid Body Motion. Cont'd

Let us take as a pole the body mass center, that is the point whose radius vector $\mathbf{r}_0$ satisfies the relation

$$\iiint\limits_{v_{\mathcal{P}}} \rho(\mathbf{r} - \mathbf{r}_0)\, dv = \mathbf{0}.$$

Then the momentum and the moment of momentum of the rigid body take the form

$$\mathfrak{P} = m\mathbf{v}_0, \qquad \mathfrak{M} = \iiint\limits_{v_{\mathcal{P}}} \rho \mathbf{z} \times \dot{\mathbf{z}}\, dv = \iiint\limits_{v_{\mathcal{P}}} \rho \mathbf{z} \times (\boldsymbol{\omega} \times \mathbf{z})\, dv = \mathbf{J} \cdot \boldsymbol{\omega}, \quad (13)$$

where $\mathbf{v}_0 = \dot{\mathbf{u}}$ and $\mathbf{J}$ is the inertia tensor:

$$\mathbf{J} \triangleq \iiint\limits_{v_{\mathcal{P}}} \rho\big[(\mathbf{z} \cdot \mathbf{z})\mathbf{I} - \mathbf{z} \otimes \mathbf{z}\big]\, dv. \quad (14)$$

Introduction: Inertia Tensor.

It is seen that $\mathbf{J}$ possesses the following property

$$\mathbf{J} = \mathbf{Q} \cdot \mathbf{J}_0 \cdot \mathbf{Q}^T, \qquad \mathbf{J}_0 \triangleq \iiint\limits_{V_{\mathcal{P}}} \rho[(\mathbf{Z} \cdot \mathbf{Z})\mathbf{I} - \mathbf{Z} \otimes \mathbf{Z}]\, dv, \tag{15}$$

where the volume integral is taken over $V_{\mathcal{P}}$ in the initial body configuration. The constant tensor $\mathbf{J}_0$ can be called the inertia tensor in the initial configuration. For example, for a homogeneous ball of radius a

$$\mathbf{J} = \frac{2}{5} m a^2 \mathbf{I} = \mathbf{J}_0.$$

If the directors $\mathbf{d}_k$ are the unit vectors along the principle axes then

$$\mathbf{J} = J_x \mathbf{d}_1 \otimes \mathbf{d}_1 + J_y \mathbf{d}_2 \otimes \mathbf{d}_2 + J_z \mathbf{d}_3 \otimes \mathbf{d}_3, \quad \mathbf{J}_0 = J_x \mathbf{D}_1 \otimes \mathbf{D}_1 + J_y \mathbf{D}_2 \otimes \mathbf{D}_2 + J_z \mathbf{D}_3 \otimes \mathbf{D}_3,$$

where J_x, J_y, J_z are moments of inertia with respect to the principal axes. The derivative of $\mathbf{J}$ satisfies the relation

$$\dot{\mathbf{J}} = \boldsymbol{\omega} \times \mathbf{J} - \mathbf{J} \times \boldsymbol{\omega}. \tag{16}$$

Introduction: Euler's motion laws

The rigid body motion is described by two Euler's motion laws.

1. The velocity change of the rigid body momentum is equal to the resulting vector of forces $\mathfrak{F}$*, acting on the body:*

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathfrak{P} = \mathfrak{F}, \qquad \mathfrak{F} \triangleq \iiint\limits_{v_{\mathcal{P}}} \rho \mathbf{f}\, dv. \tag{17}$$

2. The velocity change of the rigid body moment of momentum with respect to pole $\mathbf{r}_0$ *is equal to the resulting moment of all the forces with respect to the pole:*

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathfrak{M} = \mathfrak{C}, \qquad \mathfrak{C} \triangleq \iiint\limits_{v_{\mathcal{P}}} \rho[(\mathbf{r}-\mathbf{r}_0)\times\mathbf{f}+\boldsymbol{\mu}]\, dv. \tag{18}$$

Here $\mathbf{f}$ and $\boldsymbol{\mu}$ are the densities of the forces and the moments.

Introduction: Rigid Body Dynamics. Conclusions.

In equilibrium, these laws reduce to

$$\mathfrak{F} = \mathbf{0}, \qquad \mathfrak{C} = \mathbf{0}. \tag{19}$$

In mechanics, Euler's laws are known since ancient time. For example, equation (19) was formulated as the lever law by Archimedes in the 3rd century BC. Euler showed that the second dynamic law, or the second equilibrium condition, is independent of the first one. From (18) and (19) we can derive Newton's laws for the system of mass points. As was mentioned in [1], to derive Euler's equations from Newton's law, we should introduce additional assumptions on the nature of the interaction between the points. Being more general, equations (18) and (19) constitute the foundation of classic mechanics as well as of continuum mechanics.

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[1] Truesdell, C., Toupin, R.: The classical field theories. In: S. Flügge (ed.) Handbuch der Physik, Vol. III/1, pp. 226–793. Springer, Berlin (1960)

Micropolar Kinematics

Each material particle $X \in \mathscr{B}$ of the polar-elastic continuum has six degrees of freedom of rigid body. The finite displacement of the polar-elastic continuum can be described by the following two smooth mappings (see Fig. 3):

$$\mathbf{r} = \chi(\mathbf{R}) = \mathbf{R} + \mathbf{u}(\mathbf{R}), \qquad \mathbf{d}_a = \mathbf{Q}(\mathbf{R}) \cdot \mathbf{D}_a, \tag{20}$$

where $\mathbf{u} \in E$ is the translation vector and $\mathbf{Q} = \mathbf{d}_a \otimes \mathbf{D}_a \in Orth^+$ is the proper orthogonal microrotation tensor, $\mathbf{Q}^{-1} = \mathbf{Q}^T$, $\det \mathbf{Q} = +1$. Two independent fields $\mathbf{u}(\mathbf{R})$ and $\mathbf{Q}(\mathbf{R})$ describe translational and rotational degrees of freedom of the polar-elastic continuum.

Micropolar Kinematics. Cont’d

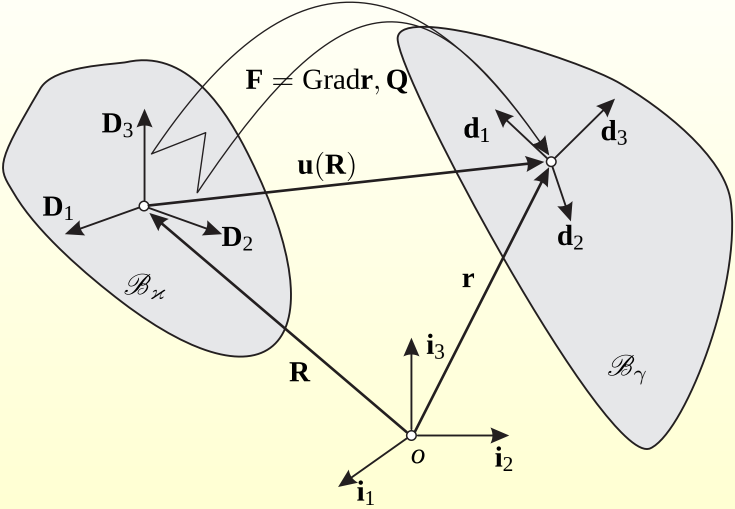


Figure: Micropolar body deformation.

Micropolar Kinematics. Velocities

The linear velocity is given by the relation

$$\mathbf{v} = \dot{\mathbf{r}}. \tag{21}$$

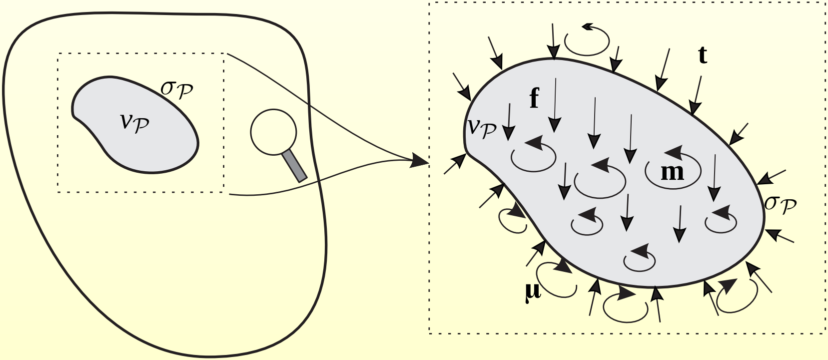
For brevity, we write out $\dot{(...)} \equiv \frac{\mathrm{d}}{\mathrm{d}\,t}(...)$, where $\frac{d}{dt}$ denotes the material derivative with respect to t. As in classical mechanics, see (11), the angular velocity vector, called microgyration vector, is given by

$$\boldsymbol{\omega} = -\frac{1}{2}\left(\mathbf{Q}^T \cdot \dot{\mathbf{Q}}\right)_\times, \tag{22}$$

where the dot denotes the dot (inner) product and $(\ldots)^T$ - transposed. Symbol $(...)_\times$ stands for the vector invariant of a second-order tensor. In particular, for a dyad $\mathbf{a} \otimes \mathbf{b}$ we have $(\mathbf{a} \otimes \mathbf{b})_\times = \mathbf{a} \times \mathbf{b}$, where $\times$ is the vector (cross) product. Relation (22) means that $\boldsymbol{\omega}$ is the axial vector associated with the skew-symmetric tensor $\mathbf{Q}^T \cdot \dot{\mathbf{Q}}$.

Forces and Couples

Forces and *couples* are the primary quantities of continuum mechanics. We assume that there are two kinds of forces and couples acting on $\mathcal{P}$, i.e. *body loads* and *contact loads*. In the case of micropolar continua o as in General Mechanics we have to take into account forces and couples as primary variables.



Forces and Couples. Cont'd

Forces and couples acting on $\mathcal{P}$ can be represented as follows

$$\mathbf{f}(\mathcal{P}) = \mathbf{f}_B(\mathcal{P}) + \mathbf{f}_C(\mathcal{P}), \quad \mathbf{m}(\mathcal{P}) = \mathbf{m}_B(\mathcal{P}) + \mathbf{m}_C(\mathcal{P}),$$

where the subscript B denotes body forces and couples and C is for the surface quantities. We introduce their mass and surface densities:

$$\mathbf{f}_B(\mathcal{P}) = \iiint\limits_{v_{\mathcal{P}}} \rho \mathbf{f} \,\mathrm{d}\, v, \quad \mathbf{m}_B(\mathcal{P}) = \iiint\limits_{v_{\mathcal{P}}} \rho \mathbf{m} \,\mathrm{d}\, v,$$

$$\mathbf{f}_C(\mathcal{P}) = \iint\limits_{\sigma_{\mathcal{P}}} \mathbf{t} \,\mathrm{d}\, \sigma, \quad \mathbf{m}_C(\mathcal{P}) = \iint\limits_{\sigma_{\mathcal{P}}} \boldsymbol{\mu} \,\mathrm{d}\, \sigma,$$

where $v_{\mathcal{P}}$ is the volume of $\mathcal{P}$ in actual configuration, $\sigma_{\mathcal{P}} = \partial v_{\mathcal{P}}$ is the boundary of $\mathcal{P}$, $\mathbf{t}$, $\boldsymbol{\mu}$ are the force and couple per the area unit in the actual configuration.

Definition

$\mathbf{t}$ *is called the stress vector and* $\boldsymbol{\mu}$ *is called the couple stress vector.*

Momentum and Moment of Momentum

Definition

Momentum of part $\mathcal{P}$ of the body is $\mathfrak{P}(\mathcal{P}) \triangleq \iiint\limits_{v_{\mathcal{P}}} \rho \mathbf{v} \, \mathrm{d}\, v.$

Definition

Moment of momentum of part $\mathcal{P}$ of the body is

$$\mathfrak{M}(\mathcal{P}) \triangleq \iiint\limits_{v_{\mathcal{P}}} \{(\mathbf{r} - \mathbf{r}_0) \times \rho \mathbf{v} + j\boldsymbol{\omega}\} \, \mathrm{d}\, v, \tag{23}$$

where ρ is the material density, $\mathbf{r}$ is the position vector in an actual configuration, $\mathbf{r}_0$ is an arbitrary position vector that does not depend on t, j is the scalar measure of rotatory inertia of “microparticles” of the material.

Eulerian Dynamic Laws

Balance of momentum. First Eulerian dynamic law. *The time rate of change of the momentum of an arbitrary part $\mathcal{P}$ of the body is equal to the total force acting on $\mathcal{P}$:*

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathfrak{P}(\mathcal{P}) = \iiint\limits_{v_{\mathcal{P}}} \rho\mathbf{f}\,\mathrm{d}\,v + \iint\limits_{\sigma_{\mathcal{P}}} \mathbf{t}\,\mathrm{d}\,\sigma. \tag{24}$$

Balance of moment of momentum. Second Eulerian dynamic law. *The time rate of change of the moment of momentum of an arbitrary part $\mathcal{P}$ of the body about a fixed point $\mathbf{r}_0$ is equal to the total moment about $\mathbf{r}_0$ acting on $\mathcal{P}$:*

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathfrak{M}(\mathcal{P}) = \iiint\limits_{v_{\mathcal{P}}} \{(\mathbf{r}-\mathbf{r}_0)\times\rho\mathbf{f} + \rho\mathbf{m}\}\,\mathrm{d}\,v + \iint\limits_{\sigma_{\mathcal{P}}} \{(\mathbf{r}-\mathbf{r}_0)\times\mathbf{t} + \boldsymbol{\mu}\}\,\mathrm{d}\,\sigma.$$

Stress Tensor and Couple Stress Tensor

Cauchy's postulate. *At a point of the body, the stress vector and the couple stress vector depend on the surface only through the unit normal to the considered surface, that is they have the same values respectively for all surfaces through the point which have the same normal.*

In what follows, the normal $\mathbf{n}$ to the body surface will be taken external with respect to the part of the body under consideration.

Cauchy's lemma. *The stress and couple stress vectors are odd functions with respect to* $\mathbf{n}$:

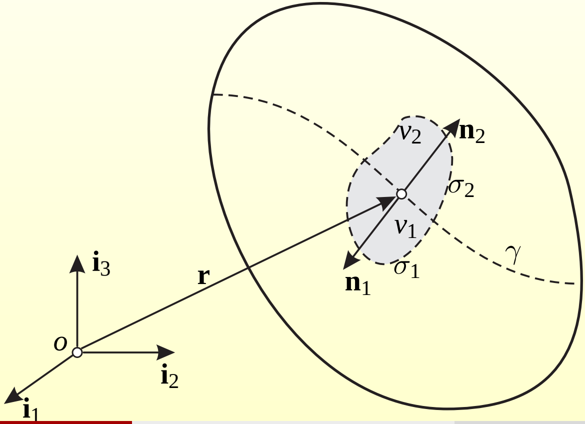
$$\mathbf{t}(\mathbf{r},\mathbf{n}) = -\mathbf{t}(\mathbf{r},-\mathbf{n}), \tag{26}$$

$$\boldsymbol{\mu}(\mathbf{r},\mathbf{n}) = -\boldsymbol{\mu}(\mathbf{r},-\mathbf{n}). \tag{27}$$

Cauchy's lemma is the re-formulation of the third Newton's axiom of reciprocal actions in micropolar continuum.

Cauchy's Lemma. Proof

Consider an arbitrary part of the body occupying domain $v_{\mathcal{P}}$, cf. Fig. 5. Surface γ through $\mathbf{r}$ splits $v_{\mathcal{P}}$ into parts v_1 and v_2, $v_{\mathcal{P}} = v_1 \cup v_2$. Note that here v_1, v_2, and γ are also arbitrary. To prove relations (26) and (27) we use Euler's dynamics laws for domains $v_{\mathcal{P}}$, v_1, and v_2.



Cauchy’s Lemma. Proof. Cont’d

For a sufficiently smooth velocity field it is valid

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathfrak{P}(\mathcal{P}) = \iiint\limits_{v_{\mathcal{P}}} \rho\frac{\mathrm{d}}{\mathrm{d}\,t}\mathbf{v}\mathrm{d}\,v.$$

Then the application of the first Euler dynamic law (24) for $v_{\mathcal{P}}$ implies

$$\iiint\limits_{v_{\mathcal{P}}} \rho\frac{\mathrm{d}}{\mathrm{d}\,t}\mathbf{v}\mathrm{d}\,v = \iiint\limits_{v_{\mathcal{P}}} \rho\mathbf{f}\mathrm{d}\,v + \iint\limits_{\sigma_{\mathcal{P}}} \mathbf{t}\mathrm{d}\,\sigma. \tag{28}$$

Applying (24) on v_1 we get

$$\iiint\limits_{v_1} \rho\frac{\mathrm{d}}{\mathrm{d}\,t}\mathbf{v}\mathrm{d}\,v = \iiint\limits_{v_1} \rho\mathbf{f}\mathrm{d}\,v + \iint\limits_{\sigma_1} \mathbf{t}\mathrm{d}\,\sigma + \iint\limits_{\gamma} \mathbf{t}(\mathbf{n}_1)\mathrm{d}\,\sigma, \tag{29}$$

where σ_1 is a part of σ that belongs to v_1 and $\mathbf{n}_1$ is the external unit normal to γ.

Cauchy’s Lemma. Proof. Cont’d

Similarly, for v_2 we have

$$\iiint\limits_{v_2} \rho \frac{\mathrm{d}}{\mathrm{d}\,t}\mathbf{v}\mathrm{d}\,v = \iiint\limits_{v_2} \rho\mathbf{f}\mathrm{d}\,v + \iint\limits_{\sigma_2} \mathbf{t}\mathrm{d}\,\sigma + \iint\limits_{\gamma} \mathbf{t}(\mathbf{n}_2)\mathrm{d}\,\sigma. \tag{30}$$

Subtracting (29), (30) from equality (28) we get

$$\mathbf{0} = \iint\limits_{\gamma} \mathbf{t}(\mathbf{n}_1)\mathrm{d}\,\sigma + \iint\limits_{\gamma} \mathbf{t}(\mathbf{n}_2)\mathrm{d}\,\sigma. \tag{31}$$

Supposing the integrands in (31) to be sufficiently smooth we derive

$$\mathbf{t}(\mathbf{n}_1) + \mathbf{t}(\mathbf{n}_2) = \mathbf{0}.$$

As $\mathbf{n}_1 = -\mathbf{n}_2$, it follows equality (26) that completes the proof of the first part of the lemma.

Cauchy’s Lemma. Proof. Cont’d

The proof of equality (27) is similar. It holds

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathfrak{M}(\mathcal{P}) = \iiint\limits_{v_{\mathcal{P}}} \left\{ (\mathbf{r}-\mathbf{r}_0) \times \rho \frac{\mathrm{d}}{\mathrm{d}\,t}\mathbf{v} + j\frac{\mathrm{d}}{\mathrm{d}\,t}\boldsymbol{\omega} \right\} \mathrm{d}\,v.$$

Applying second dynamic law (25) for domains $v_{\mathcal{P}}$, v_1, v_2 and repeating the transformations we get

$$\mathbf{0} = \iint\limits_{\gamma} \{(\mathbf{r}-\mathbf{r}_0) \times (\mathbf{t}(\mathbf{n}_1) + \mathbf{t}(\mathbf{n}_2)) + \boldsymbol{\mu}(\mathbf{n}_1) + \boldsymbol{\mu}(\mathbf{n}_2)\} \,\mathrm{d}\,\sigma. \tag{32}$$

Using (26) from (32) it follows the second equality (27) that completes the proof of Cauchy’s lemma.

Cauchy’s Theorem

Cauchy’s lemma is an essential tool for the introduction of the stress tensor and the couple stress tensor that are presented by *Cauchy’s theorem*.

Theorem

For any point of the body there exist second-order tensors $\mathbf{T}$ *and* $\mathbf{M}$ *such that the stress vector and the couple stress vector acting at a point of the surface with normal* $\mathbf{n}$ *are presented by formulae*

$$\mathbf{t} = \mathbf{T} \cdot \mathbf{n}, \quad \boldsymbol{\mu} = \mathbf{M} \cdot \mathbf{n}.$$

Proof of Cauchy’s Theorem

First let us prove the existence of $\mathbf{T}$. Consider an arbitrary orthogonal parallelepiped Π with edges oriented along the axes of Cartesian coordinates x_1, x_2, x_3, cf. Fig. 6.

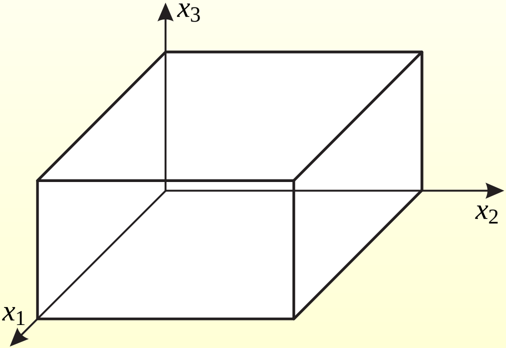


Figure: Parallelepiped Π

Proof of Cauchy’s Theorem. Cont’d

The unit normal to a side of the parallelepiped coincides with one one of the coordinate unit vectors $\mathbf{n} = \pm\mathbf{i}_k$ $(k = 1, 2, 3)$ up to direction. In frame $\mathbf{i}_k$, the stress vector is

$$\mathbf{t}(\mathbf{r}, \mathbf{Q}, \mathbf{i}_k) = t_{sk}(\mathbf{r}, \mathbf{Q})\mathbf{i}_s. \tag{33}$$

In the rest part of the proof we will omit arguments $\mathbf{r}$ and $\mathbf{Q}$. Here t_{sk} are the components of $\mathbf{t}$ in the frame $\mathbf{i}_k$. From the law of reciprocal actions it follows

$$\mathbf{t}(-\mathbf{i}_k) = -t_{sk}\mathbf{i}_s.$$

We denote n_k the components of the normal in frame $\mathbf{i}_k$. For coordinate area elements, the representation (33) is

$$\mathbf{t}(\mathbf{i}_1) = n_1 t_{s1}\mathbf{i}_s, \quad \mathbf{t}(\mathbf{i}_2) = n_2 t_{s2}\mathbf{i}_s, \quad \mathbf{t}(\mathbf{i}_3) = n_3 t_{s3}\mathbf{i}_s. \tag{34}$$

Proof of Cauchy’s Theorem. Cont’d

Let us apply the first dynamic law to the parallelepiped. We get

$$\iiint\limits_{v_\Pi} \rho \left(\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t} - \mathbf{f} \right) \mathrm{d}\,v = \iint\limits_{\sigma_\Pi} \mathbf{t} \mathrm{d}\,\sigma.$$

Using (34) the latter equation can be written as

$$\iiint\limits_{v_\Pi} \rho \left(\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t} - \mathbf{f} \right) \mathrm{d}\,v = \iint\limits_{\sigma_\Pi} n_k t_{sk} \mathbf{i}_s \mathrm{d}\,\sigma.$$

Applying Gauss–Ostrogradsky’s theorem to the surface integral we get

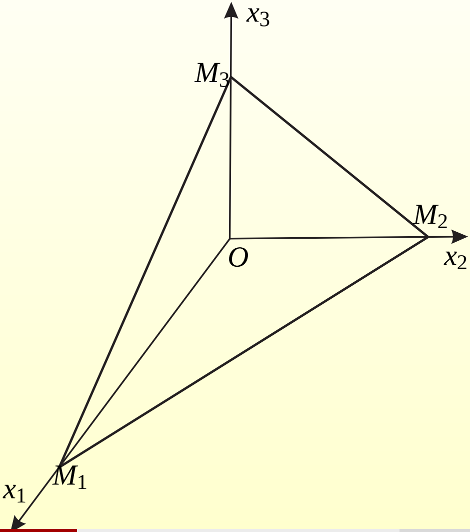
$$\iiint\limits_{v_\Pi} \left\{ \rho \left(\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t} - \mathbf{f} \right) - \frac{\partial t_{sk}}{\partial x_k} \mathbf{i}_s \right\} \mathrm{d}\,v = \mathbf{0}. \tag{35}$$

As the parallelepiped is arbitrary, from (35) it follows

$$\rho \left(\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t} - \mathbf{f} \right) - \frac{\partial t_{sk}}{\partial x_k} \mathbf{i}_s = \mathbf{0}. \tag{36}$$

Proof of Cauchy’s Theorem. Cont’d

Let us consider now an arbitrary tetrahedron T represented on Fig. 7.



Proof of Cauchy’s Theorem. Cont’d

Applying the first dynamic law for T we get

$$\iiint\limits_{v_T} \rho \left(\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t} - \mathbf{f} \right) \mathrm{d}\,v = \iint\limits_{\sigma_T} n_k t_{sk} \mathbf{i}_s \mathrm{d}\,\sigma + \iint\limits_{M_1M_2M_3} \mathbf{t}(\mathbf{n}) \mathrm{d}\,\sigma, \qquad (37)$$

where v_T is the volume, σ_T is the part of the tetrahedron sides that are parallel to the coordinate planes, $M_1M_2M_3$ is the inclined side. In (37) we used the representation of the stress vector on the coordinate plane elements (34). With the help of (36) we transform the left side of (37)

$$\iiint\limits_{v_T} \frac{\partial t_{sk}}{\partial x_k} \mathbf{i}_s \mathrm{d}\,v = \iint\limits_{\sigma_T} n_k t_{sk} \mathbf{i}_s \mathrm{d}\,\sigma + \iint\limits_{M_1M_2M_3} \mathbf{t}(\mathbf{n}) \mathrm{d}\,\sigma. \qquad (38)$$

Applying again Gauss–Ostrogradsky’s formula to the volume integral in (38) we obtain

$$\iint\limits_{M_1M_2M_3} \left[\mathbf{t}(\mathbf{n}) - n_k t_{sk} \mathbf{i}_s \right] \mathrm{d}\,\sigma = \mathbf{0}.$$

Proof of Cauchy's Theorem. Cont'd

As the tetrahedron and its side $M_1M_2M_3$ are arbitrary it follows the identity

$$\mathbf{t}(\mathbf{n}) - n_k t_{sk} \mathbf{i}_s = \mathbf{0}$$

that is valid for any normal $\mathbf{n}$. Thus, we have demonstrated that $\mathbf{t}$ depends on $\mathbf{n}$ linearly.

It is well known the following representation theorem of a linear function, see for example Lebedev et al. (2010),

Theorem

A linear vector-valued function $\mathbf{l}(\mathbf{n})$ of a vectorial argument $\mathbf{n}$ can be represented in the form $\mathbf{l}(\mathbf{n}) = \mathbf{L} \cdot \mathbf{n}$, where $\mathbf{L}$ is a second-order tensor.

By this theorem, there exists $\mathbf{T}$ such that

$$\mathbf{t}(\mathbf{n}) = \mathbf{T} \cdot \mathbf{n}. \tag{39}$$

Thus the first part of Cauchy's theorem is proved.

Proof of Cauchy’s Theorem. Cont’d

The second theorem part on the existence of $\mathbf{M}$ is proven similarly. First we should apply the second dynamic Euler’s law first to an arbitrary parallelepiped and then to an arbitrary tetrahedron.
Let us represent the couple stress vector $\boldsymbol{\mu}$ on the coordinate plane elements as follows

$$\boldsymbol{\mu}(\mathbf{i}_k) = m_{sk}\mathbf{i}_s.$$

With the use of the components of the unit normal on the coordinate plane elements it can be written as

$$\boldsymbol{\mu}(\mathbf{i}_1) = n_1 m_{s1}\mathbf{i}_s, \quad \boldsymbol{\mu}(\mathbf{i}_2) = n_2 m_{s2}\mathbf{i}_s, \quad \boldsymbol{\mu}(\mathbf{i}_3) = n_3 m_{s3}\mathbf{i}_s. \tag{40}$$

Here we used the equality $\boldsymbol{\mu}(-\mathbf{i}_k) = -m_{sk}\mathbf{i}_s$.

Proof of Cauchy’s Theorem. Cont’d

Applying the second dynamic law to Π we get

$$\iiint\limits_{v_\Pi} \left\{(\mathbf{r}-\mathbf{r}_0)\times\rho\left(\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t}-\mathbf{f}\right)+j\frac{\mathrm{d}\,\boldsymbol{\omega}}{\mathrm{d}\,t}-\rho\mathbf{m}\right\}\mathrm{d}\,v \tag{41}$$

$$=\iint\limits_{\sigma_\Pi}\{(\mathbf{r}-\mathbf{r}_0)\times\mathbf{t}+\boldsymbol{\mu}\}\,\mathrm{d}\,\sigma.$$

Using (34) and (40), we rewrite the surface integral in (41) as follows

$$\iint\limits_{\sigma_\Pi}\{(\mathbf{r}-\mathbf{r}_0)\times n_k t_{sk}\mathbf{i}_s+n_k m_{sk}\mathbf{i}_s\}\,\mathrm{d}\,\sigma.$$

Applying Gauss–Ostrogradsky’s theorem, we transform the integral over σ_Π to the integral over volume v_Π

$$\iiint\limits_{v_\Pi}\frac{\partial}{\partial x_k}\{(\mathbf{r}-\mathbf{r}_0)\times t_{sk}\mathbf{i}_s+m_{sk}\mathbf{i}_s\}\,\mathrm{d}\,v.$$

Proof of Cauchy's Theorem. Cont'd

Thus with regard to identity $\partial \mathbf{r}/\partial x_k = \mathbf{i}_k$ we reduce Eq. (41) to the form

$$\iiint\limits_{v_\Pi} \left\{ (\mathbf{r} - \mathbf{r}_0) \times \left[\rho \left(\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t} - \mathbf{f} \right) - \frac{\partial t_{sk}}{\partial x_k} \mathbf{i}_s \right] \right.$$
$$\left. + j \frac{\mathrm{d}\,\boldsymbol{\omega}}{\mathrm{d}\,t} - \rho \mathbf{m} - t_{sk} \mathbf{i}_k \times \mathbf{i}_s - \frac{\partial m_{sk}}{\partial x_k} \mathbf{i}_s \right\} \mathrm{d}\,v = \mathbf{0}.$$

Using the first dynamic law we have shown that the expression in square brackets is equal to zero. So we get the equation

$$\iiint\limits_{v_\Pi} \left\{ j \frac{\mathrm{d}\,\boldsymbol{\omega}}{\mathrm{d}\,t} - \rho \mathbf{m} - t_{sk} \mathbf{i}_k \times \mathbf{i}_s - \frac{\partial m_{sk}}{\partial x_k} \mathbf{i}_s \right\} \mathrm{d}\,v = \mathbf{0},$$

from which it follows the differential equation

$$j \frac{\mathrm{d}\,\boldsymbol{\omega}}{\mathrm{d}\,t} - \rho \mathbf{m} = \frac{\partial m_{sk}}{\partial x_k} \mathbf{i}_s - t_{sk} \mathbf{i}_s \times \mathbf{i}_k. \tag{42}$$

Proof of Cauchy's Theorem. Cont'd

The last addendum in (42) is *the vectorial invariant* of second-order tensor $\mathbf{T}$ defined as

$$\mathbf{T}_\times \overset{\triangle}{=} t_{ks}\mathbf{i}_k \times \mathbf{i}_s.$$

Next, we will show the representation (40) is valid for any area element. Applying the second dynamic law to an arbitrary tetrahedron we get

$$\iiint\limits_{v_T} \left\{(\mathbf{r}-\mathbf{r}_0)\times\rho\left(\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t}-\mathbf{f}\right)+j\frac{\mathrm{d}\,\boldsymbol{\omega}}{\mathrm{d}\,t}-\rho\mathbf{m}\right\}\mathrm{d}\,v \tag{43}$$

$$= \iint\limits_{\sigma_T}\{(\mathbf{r}-\mathbf{r}_0)\times\mathbf{t}+n_k m_{sk}\mathbf{i}_s\}\,\mathrm{d}\,\sigma + \iint\limits_{M_1M_2M_3}\{(\mathbf{r}-\mathbf{r}_0)\times\mathbf{t}+\boldsymbol{\mu}(\mathbf{N})\}\,\mathrm{d}\,\sigma.$$

Proof of Cauchy's Theorem. Cont'd

Let us reduce (43) to an equation that contains only a surface integral over side $M_1M_2M_3$. For this, in (43) we transform the terms having factor $(\mathbf{r}-\mathbf{r}_0)$. With regard for (39) and the identity

$$\operatorname{div}\left[(\mathbf{r}-\mathbf{r}_0)\times\mathbf{T}\right]=(\mathbf{r}-\mathbf{r}_0)\times\operatorname{div}\mathbf{T}-\mathbf{T}_\times$$

we can show there holds the equality

$$\iiint\limits_{v_T}\left\{(\mathbf{r}-\mathbf{r}_0)\times\rho\left(\frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t}-\mathbf{f}\right)\right\}\mathrm{d}\,v-\iint\limits_{\partial v_T}\{(\mathbf{r}-\mathbf{r}_0)\times\mathbf{t}\}\,\mathrm{d}\,\sigma$$
$$=-\iiint\limits_{v_T}\mathbf{T}_\times\mathrm{d}\,v,$$

where $\partial v_T=\sigma_T\bigcup M_1M_2M_3$.

Proof of Cauchy's Theorem. Cont'd

Thus the volume integral in (43) takes the form

$$\iiint\limits_{v_T} \left(j \frac{\mathrm{d}\,\boldsymbol{\omega}}{\mathrm{d}\,t} - \rho \mathbf{m} + \mathbf{T}_\times \right) \mathrm{d}\,v.$$

With use of (42) we transform (43) to the necessary form

$$\iint\limits_{M_1 M_2 M_3} \{ \boldsymbol{\mu}(\mathbf{n}) - n_k m_{sk} \mathbf{i}_s(\mathbf{n}) \} \,\mathrm{d}\,\sigma = \mathbf{0}.$$

As the tetrahedron is arbitrary, this shows function $\boldsymbol{\mu}(\mathbf{n})$ is linear with respect to $\mathbf{n}$ and so there holds the representation

$$\boldsymbol{\mu}(\mathbf{n}) = \mathbf{M} \cdot \mathbf{n}. \tag{44}$$

Formula (44) completes the proof of Cauchy's theorem for Cosserat medium.

Stress and Couple-Stress Tensors of Cauchy Type

Definition

Tensor $\mathbf{T}$ is called the stress tensor of Cauchy-type and tensor $\mathbf{M}$ the couple stress tensor of Cauchy-type.

So from the theorem proof we see that matrices t_{sk} and m_{sk} represent the components of the stress tensor and of the couple stress tensor in Cartesian basis $\mathbf{i}_k$:

$$\mathbf{T} = t_{ks}\mathbf{i}_k \otimes \mathbf{i}_s, \quad \mathbf{M} = m_{ks}\mathbf{i}_k \otimes \mathbf{i}_s.$$

It should be noted an important property of the stress and couple stress tensors: they are not symmetric, in general, that is $\mathbf{T}^T \neq \mathbf{T}$, $\mathbf{M}^T \neq \mathbf{M}$. This property differs the micropolar continuum from the simple materials for which the stress tensor is always symmetric.

Stress and Couple-Stress Tensors of Cauchy Type

Non-symmetry of matrices t_{sk} and m_{sk} makes us carefully work with their indices. Let us consider an arbitrary cube in the body. The sides of the cube are oriented along the axes of Cartesian coordinate system. Tangent and normal stresses that act on the surface cube are shown on Fig. 8. Second subscript of t_{sk} means the area element with normal $\mathbf{i}_k$ whereas the first subscript shows the direction of the stress action that is $\mathbf{i}_s$.
For example, t_{13} is the stress acting on the cross-section of the body that is perpendicular to axis x_3, its action is in the direction of axis x_1.

The Signs Rule

To formulate static boundary conditions, we hold the following rule for the choice of sign for the components of the stress tensor.
The rule of signs for stress tensor. *Normal stresses components are positive if they are stretching, they are negative being compressive. A tangent stress component is positive if it acts on a plane element with the normal that coincides with the basis vector and it is co-directed with one of the basis vectors. A tangent stress component that acts on a plane element with the normal that is opposite to a basis vector, is positive if it is opposite to another basis vector.*

The Signs Rule. Cont’d

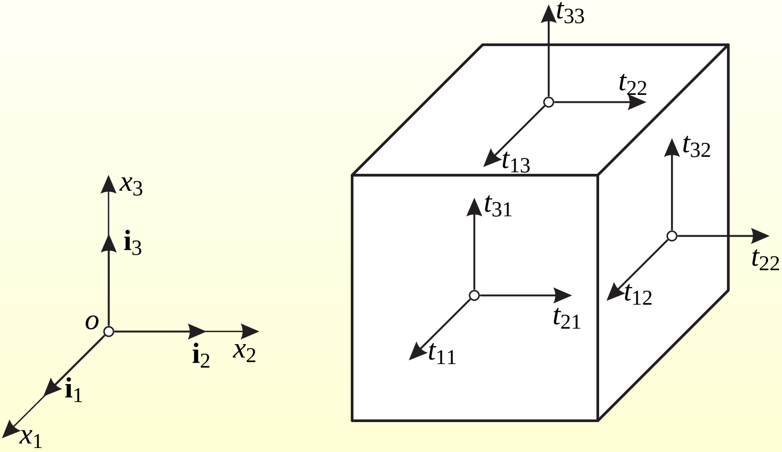


Figure: Positive stresses

The Signs Rule. Cont'd

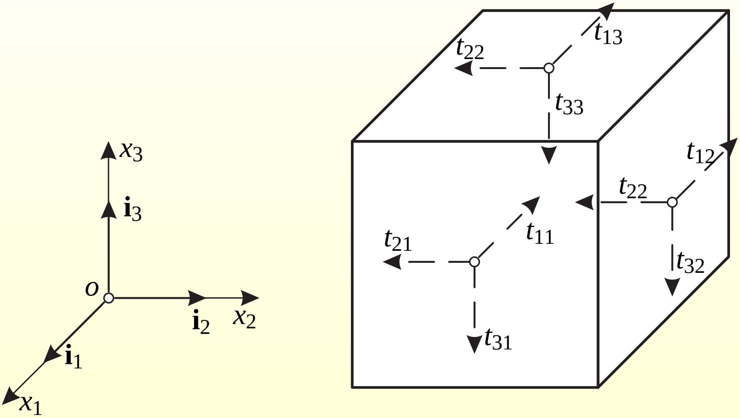


Figure: Negative stresses

Couple-stresses

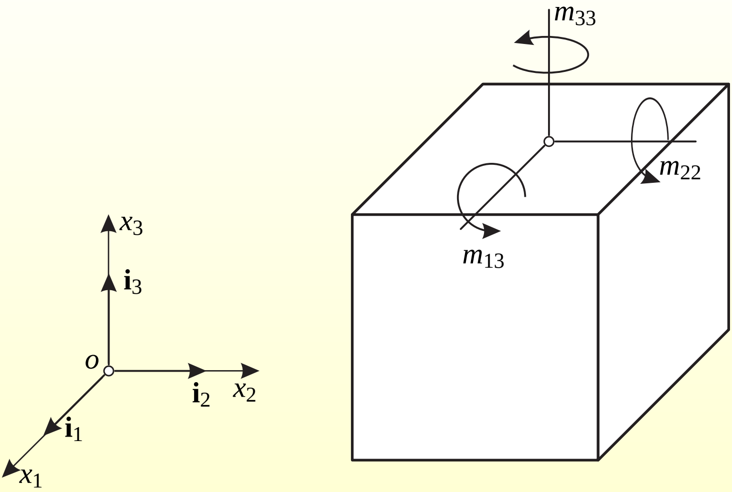


Figure: Couple stresses

Couple-stresses

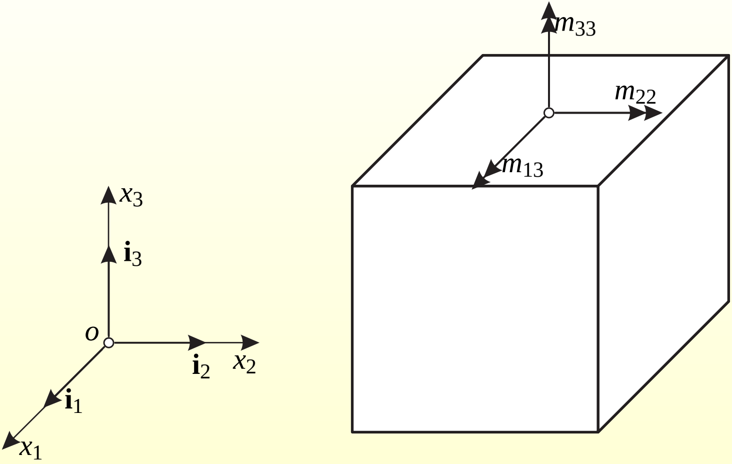
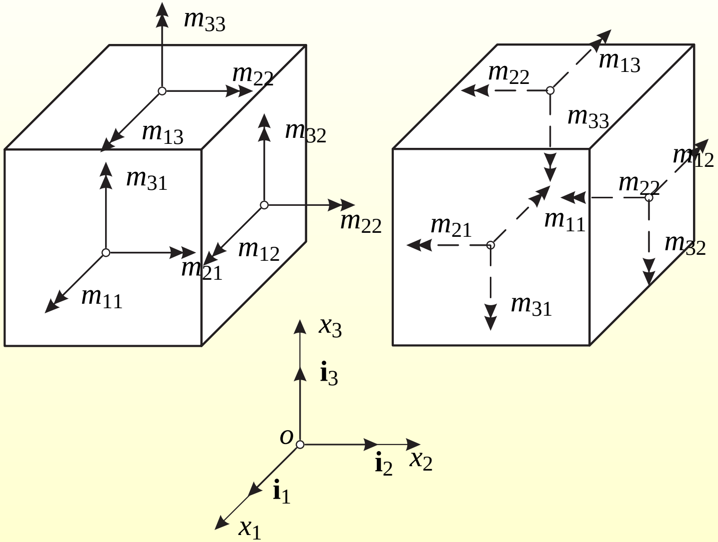


Figure: Another notation for couple stresses

The Signs Rule for Couple-stresses

The sign rule for the couple stresses. *Distributed on a plane element a torsion torque is positive when it acts in clockwise direction if to see along the external to the area element normal.*
Let a bending couple stress, a tangent couple stress, act on an area element with the normal that is co-directed with a basis vector. It is positive if it acts in the counterclockwise direction when we see in the direction that is opposite to another basis vector. For a bending couple stress on an area element with the normal that is opposite to a basis vector, the positive direction is if see the clockwise action of the couple stress when seeing in the direction that is opposite to another basis vector.

The Signs Rule for Couple-stresses



On Principal Stresses in Micropolar Continua

Representation of Cauchy stress tensor in a general non-orthogonal basis is

$$\mathbf{T} = t_{ks}\mathbf{i}_s \otimes \mathbf{i}_k,$$

where matrix t_{sk} has 9 components that are non-zero, in general.

Let us consider the problem of finding a basis in which matrix representation of $\mathbf{T}$ is most simple.

Spectral Decomposition

It is well known that for simple material with an symmetric stress tensor, the matrix of representation is diagonal

$$\mathbf{T} = \sigma_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_3 \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{45}$$

where σ_1, σ_2, σ_3 are eigenvalues of the matrix

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix},$$

that are called *principal stresses*, and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, are eigenvectors of t_{ks} that are called *principal axes* of $\mathbf{T}$. The principle axes are the normals to the surface elements on which tangential stresses are absent. As for simple material $\mathbf{T}$ is always symmetric, there exists an orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in which the matrix for $\mathbf{T}$ is diagonal. The relation (45) is called *spectral decomposition* of second-order symmetric tensor $\mathbf{T}$.

Singular Value Decomposition

In micropolar mechanics, $\mathbf{T}$ is not symmetric, in general, and so a spectral decomposition of $\mathbf{T}$ can be absent but we havethe matrix singular value decomposition[2] of $\mathbf{T}$ that exists for all non-symmetric tensors.

Definition

We call the singular value decomposition of a second-order tensor $\mathbf{T}$ *the following relation*

$$\mathbf{T} = s_1 \mathbf{e}_1 \otimes \mathbf{e}'_1 + s_2 \mathbf{e}_2 \otimes \mathbf{e}'_2 + s_3 \mathbf{e}_3 \otimes \mathbf{e}'_3, \tag{46}$$

where s_k *(*$k = 1, 2, 3$*) are non-negative real numbers, that are singular values of* $\mathbf{T}$*, and* $\mathbf{e}_k$*,* $\mathbf{e}'_j$ *are two orthonormal bases.*

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[2] Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (1985)

Real Jordan Canonical Form

Question

Let us answer the question whether there exist a more simple representation of non-symmetric $\mathbf{T}$ with only one basis that can be non-orthonormal.

Answer

Real Jordan Canonical Form

Let us recall that two matrices A and B are called similar if $A = P^{-1}BP$ for some invertible matrix P.

Real Jordan Canonical Form

A non-symmetric real valued 3×3 matrix A is similar to one of the following matrices

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{47}$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & \varepsilon \\ 0 & 0 & \lambda_2 \end{pmatrix}, \tag{48}$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & \varepsilon \\ 0 & 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \lambda_1 & \varepsilon & 0 \\ 0 & \lambda_1 & \varepsilon \\ 0 & 0 & \lambda_1 \end{pmatrix}, \tag{49}$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix} \tag{50}$$

with real valued similarity matrices.

Real Jordan Canonical Form

Which of the matrices should be selected, it depends on eigenvalues of A and their multiplicity.

Thus, a non-symmetric $\mathbf{T}$ has a representation

$$\mathbf{T} = t^{\circ}_{mn}\mathbf{e}_m \otimes \mathbf{e}_n,$$

where t°_{mn} is one of the matrices given by formulas (47)–(50). Here vectors $\mathbf{e}_m$ are not orthogonal, in general.

Obviously, all these result on the representation of $\mathbf{T}$ relate to the couple stress tensor $\mathbf{M}$, that is $\mathbf{M}$ takes form

$$\mathbf{M} = m^{\circ}_{mn}\tilde{\mathbf{e}}_m \otimes \tilde{\mathbf{e}}_n,$$

where m°_{mn} is the matrix having the structure of one of (47)–(50) and $\tilde{\mathbf{e}}_n$ is non-orthogonal basis.

The Dynamic Equations

Transforming Euler's dynamic laws as was done in the proof of Cauchy's theorem, we get dynamic equations of micropolar continuum in local form:

$$\rho \frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t} = \operatorname{div} \mathbf{T} + \rho \mathbf{f}, \tag{51}$$

$$j \frac{\mathrm{d}\,\boldsymbol{\omega}}{\mathrm{d}\,t} = \operatorname{div} \mathbf{M} - \mathbf{T}_{\times} + \rho \mathbf{m}. \tag{52}$$

Deriving equation (52) we have used equation (51).
Equations (51) and (52) in Cartesian coordinates take the form

$$\rho \frac{\mathrm{d}\,v_s}{\mathrm{d}\,t} = \frac{\partial t_{sk}}{\partial x_k} + \rho f_s, \quad j \frac{\mathrm{d}\,\omega_s}{\mathrm{d}\,t} = \frac{\partial m_{sk}}{\partial x_k} + t_{mn}\epsilon_{mns} + \rho m_s, \tag{53}$$

where $\epsilon_{mns} = -(\mathbf{i}_m \times \mathbf{i}_n) \cdot \mathbf{i}_s$.

Lagrangian Motion Equations

Lagrangian equations are

$$\mathrm{Div}\,\mathbf{T}_N^T + \rho_\varkappa \mathbf{f} = \rho_\varkappa \frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t}, \tag{54}$$

$$\mathrm{Div}\,\mathbf{M}_N^T - (\mathbf{F}\cdot\mathbf{T}_N)_\times + \rho_\varkappa \mathbf{m} = j_\varkappa \frac{\mathrm{d}\,\boldsymbol{\omega}}{\mathrm{d}\,t} \quad \text{in } B_\varkappa, \tag{55}$$

$$\mathbf{r} - \mathbf{r}^0 = \mathbf{0}, \quad \mathbf{Q} - \mathbf{Q}^0 = \mathbf{0} \quad \text{along } \partial B_{\varkappa d}. \tag{56}$$

$$\mathbf{n}\cdot\mathbf{T}_N - \mathbf{t}^0 = \mathbf{0}, \quad \mathbf{n}\cdot\mathbf{M}_N - \mathbf{m}^0 = \mathbf{0} \quad \text{along } \partial B_{\varkappa f}. \tag{57}$$

The nominal tensors $\mathbf{T}_N$, $\mathbf{M}_N$ are related to the referential tensors $\mathbf{S}$ and $\mathbf{K}$ by

$$\mathbf{T}_N = \mathbf{S}^T\cdot\mathbf{Q}^T, \qquad \mathbf{M}_N = \mathbf{K}^T\cdot\mathbf{Q}^T. \tag{58}$$

The Cauchy-type stress and couple stress tensors $\mathbf{T}$ and $\mathbf{M}$ are related to the Lagrangian-type stress measures by

$$\mathbf{T} = (\det\mathbf{F})^{-1}\mathbf{F}\cdot\mathbf{T}_N = (\det\mathbf{F})^{-1}\mathbf{F}\cdot\mathbf{S}^T\cdot\mathbf{Q}^T, \tag{59}$$

$$\mathbf{M} = (\det\mathbf{F})^{-1}\mathbf{F}\cdot\mathbf{M}_N = (\det\mathbf{F})^{-1}\cdot\mathbf{F}\cdot\mathbf{K}^T\cdot\mathbf{Q}^T. \tag{60}$$

Transition to the Cauchy Continuum

When the medium does not possesses couple's properties, that is rotation interaction of particles is negligible, then in equation (52) we should change to zero the following terms: the rotation inertia j, the couple stress tensor $\mathbf{M}$ and the volume couples $\mathbf{m}$. Then as a consequence of the balance of moment of momentum we obtain the following equation:

$$\mathbf{T}_\times = \mathbf{0}. \tag{61}$$

Its solution is the symmetric stress tensor, that is $\mathbf{T} = \mathbf{T}^T$. Thus when couple stresses and the distributed external couples in the balance equation of moment of momentum are absent, it follows symmetry of Cauchy stress tensor that is a property classic continuum mechanics. Clearly, now it is impossible to consider the action of the couple loads, in particular the action of the couples distributed on the border or inside the body.

- Constitutive equations in the Cosserat continuum

Constitutive Equations

For an arbitrary part of the body, equations (51) and (52) express the balance equations for the moment and the moment of momentum. These 9 scalar equations contain 18 unknown quantities that are the components of tensors $\mathbf{T}$ and $\mathbf{M}$. Dependence of $\mathbf{T}$ and $\mathbf{M}$ on deformation of the medium is determined by the *constitutive equations* (*constitutive relations*). The constitutive equations depend on the properties of a material. They should be determined from experimental data.

Besides, the constitutive equations must obey some principles that restrict the form of the relations, see[3]

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[3] Truesdell, C., Noll, W.: The nonlinear field theories of mechanics. In: S. Flügge (ed.) Handbuch der Physik, Vol. III/3, pp. 1–602. Springer, Berlin (1965)

Principles of Determinism and Localization

Principle of Determinism.

At each body point, the stress tensor as well as the couple stress tensor are uniquely determined by the pre-history of the motion of the body.

Principle of Localization.

At a point, the stress tensor and the couple stress tensor are uniquely determined by the motion of any neighborhood of the point that can be so small as is desired.

Let us note that Principle of Determinism is out of doubts, it reads the constitutive equations cannot forecast the future. However the Principle of Localization is not absolute, it can be non-valid for some materials.

Principle of Material Frame-Indifference

Axiom of Objectivity

Being determined by the constitutive equations, the stress tensor and the couple stress tensor must be indifferent quantities.

We remind tensor $\mathbf{T}$ is called indifferent if for two equivalent motions there holds

$$\mathbf{T}^* = \mathbf{O} \cdot \mathbf{T} \cdot \mathbf{O}^T,$$

where superscript $*$ denotes the quantities in the equivalent motion.
Two motions are called *equivalent* if they are related as follows

$$\mathbf{r}^* = \mathbf{a}(t) + \mathbf{O}(t) \cdot (\mathbf{r} - \mathbf{r}_0), \quad \mathbf{Q}^* = \mathbf{O}(t) \cdot \mathbf{Q},$$

where $\mathbf{O}(t)$ is an arbitrary orthogonal tensor, $\mathbf{a}(t)$ is an arbitrary vector and $\mathbf{r}_0$ is constant position vector that represent a fixed point.
We can treat equivalent motions as the only body motion that is considered in different reference frames.

Conclusion

- Kinematics of the micropolar continua is discussed
- The stress and couple stress tensors are introduced
- Various formulations of boundary value problems are discussed
- Principles of constitutive modelling are presented

Thank you for your attention!!!

Further questions:

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