

State of the Art and New Applications of the Six-Parameter Shell Theory

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Outline

- 1 Introduction
- 2 Formulation Principles
- 3 Six-Parameter Shell Theory
- 4 Relations with the Nonlinear Elasticity
- 5 Linear Theory
- 6 Constitutive Equations & Parameters

Shell and Plate Theories in Mechanics and Biology

- Shell-like and plate-like structures are used in **civil and aerospace engineering** as basic elements of constructions. Such structures are applied as a model of analysis in other branches, e.g. **mechanical engineering**, but also in new branches like **medicine**.
- New applications are primarily related to **advanced materials** - instead of steel or concrete, one has to analyze **sandwiches, laminates, foams, nano-films, biological membranes**, etc.
- The new trends in applications demand improvements of the theoretical foundations of the shell theory, since **new effects** must be taken into account. For example, in the case of small-size shell-like structures (nano-tubes) the **surface effect** plays an important role in the mechanical analysis of these structural elements.

Conferences in this Field

- Regular Conferences

- ▶ Shell Structures, Theory and Applications: Cracow (1974), Gołun (1978), Opole (1982), Szklarska Poręba (1986), Janowice (1992), Jurata (1998, 2002, 2005, 2009), Gdańsk (2013)

- EUROMECH Colloquia

- ▶ 444: Theories of Plates and Shells: Critical Review and New Applications (Bremen, Germany, 2002)
- ▶ 527: Shell-like Structures - Nonclassical Theories and Applications (Lutherstadt Wittenberg, Germany, 2011)

- IUTAM Symposium

- ▶ Relation of Shell, Plate, Beam and 3D Models (Tbilisi, Georgia, 2007)

- Several CISM Courses in Udine (Italy)

- Shell and Membrane Theories in Mechanics and Biology: from Macro- to Nanoscale Structures (Minsk, Belarus, 2013)

Basic Statements

- Shells and plates are **structural mechanics models** of a certain class of solid deformable bodies which are characterized by the statement that one dimension is much smaller in comparison with the other two dimensions.
- Such structures can be analyzed with the help of the 3D equations of continuum mechanics. But in this case as usual only numerical solution techniques (mostly based on the FEM) can be applied. Regardless of the progress in numerical methods and computer power sometimes the solution of shell and plate problems results in difficulties. For example, it is well known that if the thickness is very thin (close to zero) two phenomena occurs: the existence of boundary layers and the presence of locking ¹.





¹Manil Suri, Ivo Babuška, and Christoph Schwab. Locking effects in the finite element approximation of plate models. *Mathematics of Computation*, 64(210):461-482, 1995

Consequences

- Development of lower-dimensional theories applying the thinness hypothesis.
- If one accepts that one dimension is significant smaller instead of the three-dimensional (3D) partial differential equations two-dimensional (2D) ones can be introduced.

Approaches – Basic References

The deduction of the system of such equations are performed by different approaches.






-  P. M. Naghdi. The theory of plates and shells. In S. Flügge, editor, *Handbuch der Physik*, volume VIa/2, pp. 425–640. Springer, Heidelberg, 1972.
-  E. I. Grigolyuk and I. T. Selezov. Nonclassical theories of vibration of beams, plates and shells (in Russ.). In *Itogi nauki i tekhniki*, volume 5 of *Mekhanika tverdogo deformiruemogo tela*. VINITI, Moscow, 1973.
-  E. Reissner. Reflection on the theory of elastic plates. *Appl. Mech. Rev.*, 38(11):1453–1464, 1985.
-  H. Rotherth. Lineare konstitutive Gleichungen der viskoelastischen Cosseratfläche. *ZAMM*, 55(11):647–656, 1975.

Reduction of the Dimension

Start from the three-dimensional equations

- With the help of some **hypotheses** the 3D equations are transformed into 2D ones. This method is preferred by the engineers since the hypotheses are mostly based on the engineering assumptions on the stress, strain and the displacement state in the structure.
- By **mathematical techniques** such as expansions in power series, using of special functions, asymptotic integration, variational methods and combinations of mentioned methods, the 3D equations are transformed. The thinness hypotheses is directly used. The correctness is obvious if the mathematics in the background is applied carefully.
- With the **through-the-thickness integration procedure** of the 3D equilibrium or motion equations one can also derive the 2D equations of plates and shells.

Method of Hypotheses

-  G. Kirchhoff. Über das Gleichgewicht und die Bewegung einer elastischen Scheibe. *J. Reine Angew. Math.*, 40:51–88, 1850.
-  A. E. H. Love. On the small vibrations and deformations of thin elastic shells. *Phil. Trans. R. Soc.*, 179:491–546, 1888.
-  E. Reissner. On the theory of bending of elastic plates. *J. Math. Phys.*, 23:184–194, 1944.
-  R. D. Mindlin. Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates. *Trans. ASME. J. Appl. Mech.*, 18:31–38, 1951.
-  P. M. Naghdi. On the theory of thin elastic shells. *Quart. Appl. Math.*, 14:369–380, 1957.

Direct Approach

Start from the two-dimensional equations

In the **direct approach** a material deformable surface is introduced as a model of the shell. Since such surface should inherit the orientational degrees of freedom of the shell, which can be described by few additional vectors called directors, this model is often called the **directed surface**. Like in the classical shell theories in the theory of directed surfaces forces, moments and/or other static quantities with the corresponding dual surface strain quantities are the primary variables. At the beginning of the development of the theory of deformable surfaces 3D images are not used. In this sense the theory of deformable surfaces is the 2D analogue of the 3D continuum mechanics. All basic elements (kinematics, kinetics, balance equations and constitutive equations) are formulated in dependence of 2 space coordinates representing the surface.

Direct Approach – Historical Remarks

The direct approach in the theories of rods, plates and shells has a long history. The first contribution in this field was made by **Leonhard Euler** (using the suggestion of **Jakob I. Bernoulli** that the moment is an independent quantity). Later the **Cosserat brothers**² present a nearly complete new theory. But since they did not presented any constitutive equations the theory was not applied in practice in that time. Only few researchers discussed the problem of the direct approach in the first half of the 20th century (among them **G. Hamel**³).





²E. Cosserat and F. Cosserat. *Théorie des corps déformables*. Herman et Fils, Paris, 1909

³G. Hamel. *Theoretische Mechanik: Eine einheitliche Einführung in die gesamte Mechanik*. Springer, Berlin, 1948

Direct Approach - “Revitalization”

- The renaissance of the direct approach started in 1958 with the pioneering work of Ericksen & Truesdell.
- More information on the history of the direct approach one can find in the reviews of Naghdi, Rubin or Altenbach, Altenbach & Eremeyev.
- Let us note that in the theory of directed surfaces there are various assumptions related to the number of additional degrees of freedom. For example, the model of Ericksen & Truesdell contains $3 + 3p$ degrees of freedom, where p is the number of attached deformable directors. Zhilin introduced three rigid orthogonal directors, i.e. three translational and three rotational degrees of freedom, Eliseev & Vetyukov used three translational and two rotational degrees of freedom.

References Direct Approach

-  J. L. Ericksen and C. Truesdell. Exact theory of stress and strain in rods and shells. *Arch. Rat. Mech. Anal.*, 1(1):295–323, 1958.
-  M. B. Rubin. *Cosserat Theories: Shells, Rods and Points*. Kluwer, Dordrecht, 2000.
-  V. V. Eliseev and Yu. M. Vetyukov. Finite deformation of thin shells in the context of analytical mechanics of material surfaces. *Acta Mech.*, 209:43-57, 2010.
-  J. Altenbach, H. Altenbach, and V. A. Eremeyev. On generalized Cosserat-type theories of plates and shells: a short review and bibliography. *Arch. Appl. Mech.*, 80:73-92, 2010.





Advantages/Disadvantages

All theories have advantages and disadvantages:

- The first approaches are certain approximations of the three-dimensional continuum mechanics. This must result in some errors and may be incorrectness, in general. On the other hand the constitutive equations and the boundary conditions can be deduced using the same hypotheses or mathematical techniques.
- The direct approach is so strong and correct as the three-dimensional continuum mechanics, but special effort is necessary to deduce the constitutive equations and to identify the parameters in these equations. This is some kind of limitation for the practical use of such theories.

Our Approach

General nonlinear 6-parameter theory of shells

-  A. Libai and J. G. Simmonds. *The Nonlinear Theory of Elastic Shells*. Cambridge University Press, 1998.
-  J. Chróścielewski, J. Makowski, and W. Pietraszkiewicz. *Statics and Dynamics of Multifolded Shells. Nonlinear Theory and Finite Element Method*. IPPT PAN, Warsaw, 2004.
-  V. A. Eremeyev, L. P. Lebedev, and H. Altenbach. *Foundations of Microplar Mechanics*. Springer, Heidelberg, 2013.
-  H. Altenbach and V. A. Eremeyev. Cosserat-type shells. In *Generalized Continua from the Theory to Engineering Applications*, volume 541 of *CISM Courses and Lectures*, pages 131–178. Springer, Wien, 2013.

Notations

We use the following notations.

1. Vectors and tensors are denoted by semibold roman font like \mathbf{A} , \mathbf{a} .
2. Greece indices take values 1 and 2, Latin indices are arbitrary.
3. Grad operator.

$$\begin{aligned} [\text{Grad}\mathbf{v}(\mathbf{x})] \cdot \mathbf{a} &= \left. \frac{d}{dt} \mathbf{v}(\mathbf{x} + t\mathbf{a}) \right|_{t=0}, \\ [\text{Grad}\mathbf{A}(\mathbf{x})] \cdot \mathbf{a} &= \left. \frac{d}{dt} \mathbf{A}(\mathbf{x} + t\mathbf{a}) \right|_{t=0}, \quad \text{for any } t \in R, \mathbf{a} \in E. \end{aligned} \quad (1)$$

5. Div operator.

$$[\text{Div}\mathbf{A}(\mathbf{x})] \cdot \mathbf{a} = \text{Div}[\mathbf{A}(\mathbf{x}) \cdot \mathbf{a}], \quad \forall \mathbf{A} \in E \otimes E, \quad \forall \mathbf{a} \in E. \quad (2)$$

Introduction: Mass-points Motion

Newton's law. Motion of one and n mass-points is determined by

$$m\dot{\mathbf{v}} = \mathbf{f}, \quad m_i\dot{\mathbf{v}}_i = \mathbf{f}_i, \quad i = 1, 2, \dots, n. \quad (3)$$

Definition

The momentum of the mass-point called also the linear momentum, is the quantity $\mathfrak{P} = m\mathbf{v}$. The moment of momentum of the mass-point with respect to a point with radius vector \mathbf{r}_0 called the pole is

$$\mathfrak{M} = (\mathbf{r} - \mathbf{r}_0) \times m\mathbf{v}.$$

Definition

The momentum and the moment of momentum of n mass-points with respect pole \mathbf{r}_0 are

$$\mathfrak{P} = \sum_{i=1}^n m_i \mathbf{v}_i, \quad \text{and} \quad \mathfrak{M} = \sum_{i=1}^n (\mathbf{r}_i - \mathbf{r}_0) \times m_i \mathbf{v}_i.$$

Introduction: Mass-points Motion. Cont'd

Theorem

The rate of the change of the momentum of an n mass-points is equal to the total (resulting) force vector \mathfrak{F} , that is the sum of all the forces acting to the mass-points

$$\frac{d}{dt}\mathfrak{P} = \mathfrak{F}, \quad \mathfrak{F} \triangleq \sum_{i=1}^n \mathbf{f}_i. \quad (4)$$

Theorem

The rate of the change of the moment of momentum with respect to pole \mathbf{r}_0 of an n mass-points is equal to the total torque (resulting moment) \mathfrak{C} with respect to pole \mathbf{r}_0 of all the forces acting on the mass-points

$$\frac{d}{dt}\mathfrak{M} = \mathfrak{C}, \quad \mathfrak{C} \triangleq \sum_{i=1}^n (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{f}_i. \quad (5)$$

Introduction: Mass-points Motion. Cont'd

Theorem

The rate of the change of the momentum of an n mass-points is equal to the total (resulting) force vector \mathfrak{F} , that is the sum of all the forces acting to the mass-points

$$\frac{d}{dt}\mathfrak{P} = \mathfrak{F}, \quad \mathfrak{F} \triangleq \sum_{i=1}^n \mathbf{f}_i. \quad (6)$$

Theorem

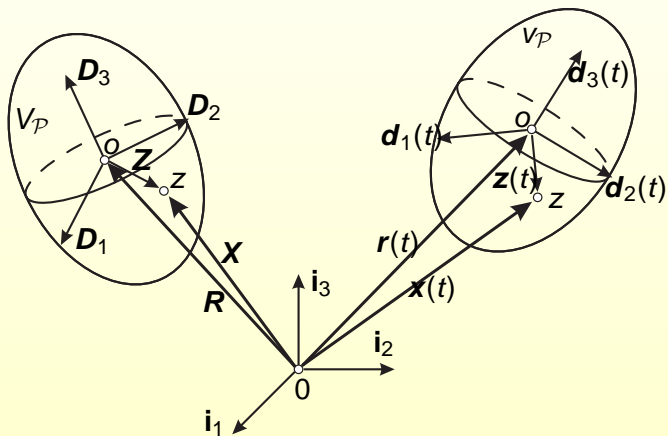
The rate of the change of the moment of momentum with respect to pole \mathbf{r}_0 of an n mass-points is equal to the total torque (resulting moment) \mathfrak{C} with respect to pole \mathbf{r}_0 of all the forces acting on the mass-points

$$\frac{d}{dt}\mathfrak{M} = \mathfrak{C}, \quad \mathfrak{C} \triangleq \sum_{i=1}^n (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{f}_i. \quad (7)$$

Introduction: Rigid Body Kinematics

Definition

A set of material points for which the mutual distances between the points remain unchanged in motion, is called a rigid body.



Introduction: Rigid Body Kinematics. Cont'd

The position any point $z \in \mathcal{P}$ is given by

$$\mathbf{r}(t) = \mathbf{R}_0 + \mathbf{u}(t) + \mathbf{Q}(t) \cdot \mathbf{Z}, \quad (8)$$

where $\mathbf{Q} = \mathbf{d}_i \otimes \mathbf{D}_i$ is the rotation tensor.

Differentiating (8) we get

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{u}}(t) + \dot{\mathbf{Q}}(t) \cdot \mathbf{Z}. \quad (9)$$

\mathbf{Q} is orthogonal so tensor $\dot{\mathbf{Q}} \cdot \mathbf{Q}^T$ is antisymmetric and it can be represented in the form

$$\dot{\mathbf{Q}} \cdot \mathbf{Q}^T = \boldsymbol{\omega} \times \mathbf{I}, \quad (10)$$

where $\boldsymbol{\omega}$ is the angular velocity of \mathcal{P} . Vector $\boldsymbol{\omega}$ can be determined as follows

$$\boldsymbol{\omega} = -\frac{1}{2}(\dot{\mathbf{Q}} \cdot \mathbf{Q}^T)_{\times} \quad (11)$$

Thus the velocity vector of a body point takes the form

$$\mathbf{v}(t) = \dot{\mathbf{u}}(t) + \boldsymbol{\omega}(t) \times \mathbf{Z}. \quad (12)$$

Introduction: Rigid Body Motion

The rigid body can be considered as a system of mass-points and so we can introduce the following definitions.

Definition

The momentum and the moment of momentum with respect to pole \mathbf{r}_0 for a rigid body are the quantities

$$\mathfrak{P} = \iiint_{V_{\mathcal{P}}} \rho \mathbf{v} \, dv, \quad \mathfrak{M} = \iiint_{V_{\mathcal{P}}} \rho (\mathbf{r} - \mathbf{r}_0) \times \mathbf{v} \, dv,$$

respectively.

Here ρ is the mass density of \mathcal{P} so its mass m is given by the integral over domain $V_{\mathcal{P}} \subset \mathbb{R}^3$ taken by the body in space, $m(\mathcal{P}) = \iiint_{V_{\mathcal{P}}} \rho \, dv$.

Introduction: Rigid Body Motion. Cont'd

Let us take as a pole the body mass center, that is the point whose radius vector \mathbf{r}_0 satisfies the relation

$$\iiint_{V_P} \rho(\mathbf{r} - \mathbf{r}_0) dv = \mathbf{0}.$$

Then the momentum and the moment of momentum of the rigid body take the form

$$\mathfrak{P} = m\mathbf{v}_0, \quad \mathfrak{M} = \iiint_{V_P} \rho \mathbf{z} \times \dot{\mathbf{z}} dv = \iiint_{V_P} \rho \mathbf{z} \times (\boldsymbol{\omega} \times \mathbf{z}) dv = \mathbf{J} \cdot \boldsymbol{\omega}, \quad (13)$$

where $\mathbf{v}_0 = \dot{\mathbf{u}}$ and \mathbf{J} is the inertia tensor:

$$\mathbf{J} \triangleq \iiint_{V_P} \rho [(\mathbf{z} \cdot \mathbf{z})\mathbf{I} - \mathbf{z} \otimes \mathbf{z}] dv. \quad (14)$$

Introduction: Inertia Tensor.

It is seen that \mathbf{J} possesses the following property

$$\mathbf{J} = \mathbf{Q} \cdot \mathbf{J}_0 \cdot \mathbf{Q}^T, \quad \mathbf{J}_0 \triangleq \iiint_{V_P} \rho [(\mathbf{Z} \cdot \mathbf{Z})\mathbf{I} - \mathbf{Z} \otimes \mathbf{Z}] dv, \quad (15)$$

where the volume integral is taken over V_P in the initial body configuration. The constant tensor \mathbf{J}_0 can be called the inertia tensor in the initial configuration. For example, for a homogeneous ball of radius a

$$\mathbf{J} = \frac{2}{5} ma^2 \mathbf{I} = \mathbf{J}_0.$$

If the directors \mathbf{d}_k are the unit vectors along the principle axes then

$$\mathbf{J} = J_x \mathbf{d}_1 \otimes \mathbf{d}_1 + J_y \mathbf{d}_2 \otimes \mathbf{d}_2 + J_z \mathbf{d}_3 \otimes \mathbf{d}_3, \quad \mathbf{J}_0 = J_x \mathbf{D}_1 \otimes \mathbf{D}_1 + J_y \mathbf{D}_2 \otimes \mathbf{D}_2 + J_z \mathbf{D}_3 \otimes \mathbf{D}_3$$

where J_x , J_y , J_z are moments of inertia with respect to the principal axes. The derivative of \mathbf{J} satisfies the relation

$$\dot{\mathbf{J}} = \boldsymbol{\omega} \times \mathbf{J} - \mathbf{J} \times \boldsymbol{\omega}. \quad (16)$$

Introduction: Euler's motion laws

The rigid body motion is described by two Euler's motion laws.

1. *The velocity change of the rigid body momentum is equal to the resulting vector of forces \mathfrak{F} , acting on the body:*

$$\frac{d}{dt}\mathfrak{P} = \mathfrak{F}, \quad \mathfrak{F} \triangleq \iiint_{V_P} \rho \mathbf{f} dv. \quad (17)$$

2. *The velocity change of the rigid body moment of momentum with respect to pole \mathbf{r}_0 is equal to the resulting moment of all the forces with respect to the pole:*

$$\frac{d}{dt}\mathfrak{M} = \mathfrak{e}, \quad \mathfrak{e} \triangleq \iiint_{V_P} \rho [(\mathbf{r} - \mathbf{r}_0) \times \mathbf{f} + \boldsymbol{\mu}] dv. \quad (18)$$

Here \mathbf{f} and $\boldsymbol{\mu}$ are the densities of the forces and the moments.

Introduction: Rigid Body Dynamics. Conclusions.





In equilibrium, these laws reduce to

$$\mathfrak{F} = \mathbf{0}, \quad \mathfrak{C} = \mathbf{0}. \quad (19)$$

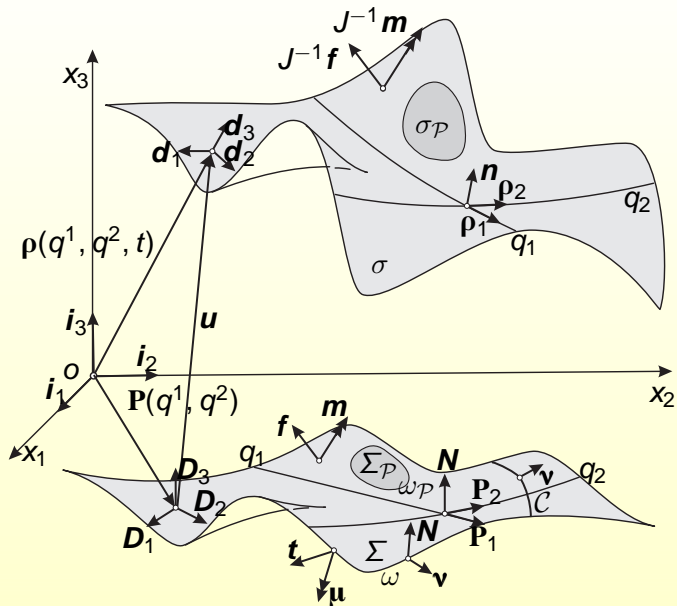
In mechanics, Euler's laws are known since ancient time. For example, equation (19) was formulated as the lever law by Archimedes in the 3rd century BC. Euler showed that the second dynamic law, or the second equilibrium condition, is independent of the first one. From (18) and (19) we can derive Newton's laws for the system of mass points. As was mentioned in ⁴, to derive Euler's equations from Newton's law, we should introduce additional assumptions on the nature of the interaction between the points. Being more general, equations (18) and (19) constitute the foundation of classic mechanics as well as of continuum mechanics.

⁴Truesdell, C., Toupin, R.: The classical field theories. In: S. Flügge (ed.) Handbuch der Physik, Vol. III/1, pp. 226–793. Springer, Berlin (1960)

Notation - Basic References

-  V. A. Eremeyev. Nonlinear micropolar shells: theory and applications. In W. Pietraszkiewicz and C. Szymczak, editors, *Shell Structures: Theory and Applications*, pages 11–18. Taylor & Francis, London, 2005.
-  V. A. Eremeyev and L. M. Zubov. *Mechanics of Elastic Shells (in Russ.)*. Nauka, Moscow, 2008
-  J. Altenbach, H. Altenbach, and V. A. Eremeyev. On generalized Cosserat-type theories of plates and shells: a short review and bibliography. *Arch. Appl. Mech.*, 80:73–92, 2010.
-  V. A. Eremeyev, L. P. Lebedev, and H. Altenbach. *Foundations of Microplar Mechanics*. Springer, Heidelberg, 2013.

Shell Kinematics



Classification

The **general theory of elastic directed surfaces** belongs to the class of **6-parametric nonlinear micropolar elastic shell theory**.

- In the actual configuration the base surface \mathcal{N} of the shell is characterized by the position vector $\rho(q^1, q^2)$ and three orthogonal unit vectors $\mathbf{d}_k(q^1, q^2)$ called directors, $k = 1, 2, 3$. Here q^1, q^2 are the coordinates on the surface.
- In the reference configuration the shell is characterized by the base surface \mathcal{M} in the same manner: by the position vector $\mathbf{P}(q^1, q^2)$ and three orthogonal vectors $\mathbf{D}_k(q^1, q^2)$.

In this sense this theory is the two-dimensional analogue to the three-dimensional Cosserat or micropolar continuum^{5,6}.

⁵C.B. Kafadar and A.C. Eringen. Micropolar media – I. The classical theory. *Int. J. Engng Sc.*, 9:271-305, 1971

⁶A.C. Eringen. *Microcontinuum Field Theory. I. Foundations and Solids*. Springer, New York, 1999

Deformation of the Shell

The both sets of directors in both configurations define the following proper orthogonal tensor \mathbf{Q} :

$$\mathbf{Q} = \mathbf{D}_k \otimes \mathbf{d}_k.$$

Thus, the deformation of the shell is given by two surface fields

$$\rho = \rho(q^1, q^2), \quad \mathbf{Q} = \mathbf{Q}(q^1, q^2).$$

Instead of \mathbf{Q} one can introduce vector of rotations⁷.

⁷W. Pietraszkiewicz and V.A. Eremeyev. On vectorially parameterized natural strain measures of the non-linear Cosserat continuum. *Int. J. Solids Struct.*, 46(11-12):2477-2480, 2009

Surface Strain Energy

For a micropolar hyper-elastic shell we can introduce a strain energy density W . With regard for the local action principle, W takes the form

$$W = W(\boldsymbol{\rho}, \nabla_s \boldsymbol{\rho}, \mathbf{Q}, \nabla_s \mathbf{Q}),$$

where

$$\nabla_s \boldsymbol{\psi} \triangleq \mathbf{P}^\alpha \otimes \frac{\partial \boldsymbol{\psi}}{\partial q^\alpha} \quad (\alpha, \beta = 1, 2), \quad \mathbf{P}^\alpha \cdot \mathbf{P}_\beta = \delta_\beta^\alpha, \quad \mathbf{P}^\alpha \cdot \mathbf{N} = 0, \quad \mathbf{P}_\beta = \frac{\partial \mathbf{P}}{\partial q^\beta}$$

Here vectors \mathbf{P}_β and \mathbf{P}^α denote the natural and reciprocal bases on Σ respectively, \mathbf{N} is the unit normal to Σ , ∇_s is the surface nabla operator on Σ , and $\boldsymbol{\psi}$ is an arbitrary differentiable tensor field given on Σ .

Surface Strain Energy

Applying the **principle of frame indifference**⁸ for a hyper-elastic shell we introduce the surface strain energy W as a function of two Lagrangian surface strain tensors \mathbf{E} and \mathbf{K} ⁹

$$W = W(\mathbf{E}, \mathbf{K}),$$

where

$$\mathbf{E} = \mathbf{F} \cdot \mathbf{Q}^T - \mathbf{A}, \quad \mathbf{K} = \frac{1}{2} \mathbf{R}^\alpha \otimes \left(\frac{\partial \mathbf{Q}}{\partial q^\alpha} \cdot \mathbf{Q}^T \right)_\times \quad (20)$$

are the **surface strain measures**.

⁸C. Truesdell and W. Noll. The nonlinear field theories of mechanics. In S. Flügge, editor, *Handbuch der Physik*, Vol. II/3, pages 1-602. Springer, Berlin, 1965

⁹V.A. Eremeyev and W. Pietraszkiewicz. Phase transitions in thermoelastic and thermoviscoelastic shells. *Arch. Mech.*, 61(1):41-67, 2009

Surface Strain Energy

The proper orthogonal tensor describing the rotation about axis \mathbf{e} for angle φ can be represented with use of Gibbs's formula. Introducing the finite rotation vector $\boldsymbol{\theta} = 2\mathbf{e} \tan \varphi/2$ we get a representation of \mathbf{Q} in the form \mathbf{v} that does not contain trigonometric functions.

$$\boldsymbol{\theta} = 2(1 + \text{tr } \mathbf{Q})^{-1} \mathbf{Q}_\times. \quad (21)$$

Using the *finite rotation vector* $\boldsymbol{\theta}$ we can express \mathbf{K} as follows

$$\mathbf{K} = \mathbf{P}^\alpha \otimes \mathbf{L}_\alpha = \frac{4}{4 + \theta^2} \nabla_s \boldsymbol{\theta} \cdot \left(\mathbf{I} + \frac{1}{2} \mathbf{I} \times \boldsymbol{\theta} \right). \quad (22)$$

The strain measures \mathbf{E} and \mathbf{K} are two-dimensional analogues of the strain measures used in 3D Cosserat continuum.

Virtual work principle

Lagrangian equilibrium equations for a micropolar shell can be derived from *the virtual work principle*

$$\delta \iint_{\Sigma} W \, d\Sigma = \delta' A, \quad (23)$$

where

$$\delta' A = \iint_{\Sigma} (\mathbf{f} \cdot \delta \boldsymbol{\rho} + \mathbf{m} \cdot \delta' \boldsymbol{\psi}) \, d\Sigma + \int_{\omega_2} \mathbf{t} \cdot \delta \boldsymbol{\rho} \, ds + \int_{\omega_4} \boldsymbol{\mu} \cdot \delta' \boldsymbol{\psi} \, ds,$$

$$\mathbf{l} \times \delta' \boldsymbol{\psi} = -\mathbf{Q}^T \cdot \delta \mathbf{Q}.$$

In Eq. (23), δ is the symbol of variation, $\delta' \boldsymbol{\psi}$ the virtual rotation vector, \mathbf{f} the surface force density distributed on Σ , and \mathbf{m} the surface couple density distributed on Σ . The quantities \mathbf{t} and $\boldsymbol{\mu}$ are the linear densities of forces and couples distributed along corresponding parts of the shell boundary ω , respectively.

Lagrangian Shell Equations

Using the formulae

$$\delta W = \frac{\partial W}{\partial \mathbf{E}} \bullet \delta \mathbf{E} + \frac{\partial W}{\partial \mathbf{K}} \bullet \delta \mathbf{K},$$

$$\delta \mathbf{E} = (\nabla_s \delta \rho) \cdot \mathbf{Q}^T + \mathbf{F} \cdot \delta \mathbf{Q}^T, \quad \delta \mathbf{K} = (\nabla_s \delta' \psi) \cdot \mathbf{Q}^T,$$

$$\delta' \psi = \frac{4}{4 + \theta^2} \left(\delta \boldsymbol{\theta} + \frac{1}{2} \boldsymbol{\theta} \times \delta \boldsymbol{\theta} \right),$$

and Eq. (23), we obtain the *Lagrangian shell equations*:

$$\nabla_s \cdot \mathbf{D} + \mathbf{f} = \mathbf{0}, \quad \nabla_s \cdot \mathbf{G} + \left[\mathbf{F}^T \cdot \mathbf{D} \right]_{\times} + \mathbf{m} = \mathbf{0}, \quad (24)$$

$$\mathbf{D} = \mathbf{P}_1 \cdot \mathbf{Q}, \quad \mathbf{G} = \mathbf{P}_2 \cdot \mathbf{Q}, \quad \mathbf{P}_1 = \frac{\partial W}{\partial \mathbf{E}}, \quad \mathbf{P}_2 = \frac{\partial W}{\partial \mathbf{K}}. \quad (25)$$

Remark:

The introduced strain and stress resultant tensors are work-conjugated.

Boundary Conditions

They are supplemented by the boundary conditions:

$$\begin{array}{ll} \text{on } \omega_1 : & \boldsymbol{\rho} = \boldsymbol{\rho}(\mathbf{s}), \\ \text{on } \omega_3 : & \mathbf{Q} = \mathbf{h}(\mathbf{s}), \quad \mathbf{h} \cdot \mathbf{h}^T = \mathbf{I}, \end{array} \quad \begin{array}{ll} \text{on } \omega_2 : & \mathbf{v} \cdot \mathbf{D} = \mathbf{t}(\mathbf{s}), \\ \text{on } \omega_4 : & \mathbf{v} \cdot \mathbf{G} = \boldsymbol{\mu}(\mathbf{s}). \end{array} \quad (26)$$

Here $\boldsymbol{\rho}(\mathbf{s})$, $\mathbf{h}(\mathbf{s})$ are given vector and tensor functions, and \mathbf{v} is the external unit normal to the boundary curve ω ($\mathbf{v} \cdot \mathbf{N} = 0$). Equations (24) are the equilibrium equations for the linear momentum and angular momentum at any shell point. \mathbf{D} and \mathbf{G} are *the surface stress and couple stress tensors* of the 1st Piola-Kirchhoff type and the stress measures \mathbf{P}_1 and \mathbf{P}_2 in Eqs. (24) are the referential stress and couple stress tensors, respectively. The strain measures \mathbf{E} and \mathbf{K} are work-conjugate to the stress measures \mathbf{D} and \mathbf{G} . The boundary ω of Σ is divided into two parts in such a way that $\omega = \omega_1 \cup \omega_2 = \omega_3 \cup \omega_4$.

Eulerian form of equilibrium equations

The *equilibrium equations* may be transformed to the *Eulerian form*

$$\tilde{\nabla}_s \cdot \mathbf{T} + J^{-1} \mathbf{f} = \mathbf{0}, \quad \tilde{\nabla}_s \cdot \mathbf{M} + \mathbf{T}_x + J^{-1} \mathbf{m} = \mathbf{0}, \quad (27)$$

where

$$\tilde{\nabla}_s \cdot \psi \triangleq \rho^\alpha \cdot \frac{\partial \psi}{\partial q^\alpha}, \quad \rho^\alpha \cdot \rho_\beta = \delta_\beta^\alpha, \quad \rho^\alpha \cdot \mathbf{n} = 0, \quad \rho_\beta = \frac{\partial \rho}{\partial q^\beta},$$
$$\mathbf{T} = J^{-1} \mathbf{F}^T \cdot \mathbf{D}, \quad \mathbf{M} = J^{-1} \mathbf{F}^T \cdot \mathbf{G}, \quad (28)$$

$$J = \sqrt{\frac{1}{2} \left\{ [\text{tr} (\mathbf{F} \cdot \mathbf{F}^T)]^2 - \text{tr} [(\mathbf{F} \cdot \mathbf{F}^T)^2] \right\}}.$$

Here \mathbf{T} and \mathbf{M} are Cauchy-type surface stress and couple stress tensors, $\tilde{\nabla}_s$ is the surface nabla operator on σ related with ∇_s by the formula $\nabla_s = \mathbf{F} \cdot \tilde{\nabla}_s$, and \mathbf{n} is the unit normal to σ .

Motion equations

The *motion equations* are

$$\nabla_s \cdot \mathbf{D} + \mathbf{f} = \rho \frac{d\mathbf{K}_1}{dt}, \quad \nabla_s \cdot \mathbf{G} + [\mathbf{F}^T \cdot \mathbf{D}]_{\times} + \mathbf{m} = \rho \left(\frac{d\mathbf{K}_2}{dt} + \mathbf{v} \times \Theta_1^T \cdot \boldsymbol{\omega} \right) \quad (29)$$

where

$$\mathbf{K}_1 \triangleq \frac{\partial K}{\partial \mathbf{v}} = \mathbf{v} + \Theta_1^T \cdot \boldsymbol{\omega}, \quad \mathbf{K}_2 \triangleq \frac{\partial K}{\partial \boldsymbol{\omega}} = \Theta_1 \cdot \mathbf{v} + \Theta_2 \cdot \boldsymbol{\omega}, \quad (30)$$

$$K(\mathbf{v}, \boldsymbol{\omega}) = \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \boldsymbol{\omega} \cdot \Theta_1 \cdot \mathbf{v} + \frac{1}{2} \boldsymbol{\omega} \cdot \Theta_2 \cdot \boldsymbol{\omega}. \quad (31)$$

Here

$$\mathbf{v} = \frac{d\rho}{dt}, \quad \boldsymbol{\omega} = \frac{1}{2} \left(\mathbf{Q}^T \cdot \frac{d\mathbf{Q}}{dt} \right)_{\times}.$$

For the dynamic problem (29), the initial conditions are

$$\rho|_{t=0} = \rho^{\circ}, \quad \mathbf{v}|_{t=0} = \mathbf{v}^{\circ}, \quad \mathbf{Q}|_{t=0} = \mathbf{Q}^{\circ}, \quad \boldsymbol{\omega}|_{t=0} = \boldsymbol{\omega}^{\circ},$$

with given initial values ρ° , \mathbf{v}° , \mathbf{Q}° , $\boldsymbol{\omega}^{\circ}$.

Euler's motion laws

Let us define the momentum \mathfrak{P} and moment of momentum \mathfrak{M} of a shell part \mathcal{P} as follows

$$\mathfrak{P}(\mathcal{P}) \triangleq \iint_{\Sigma_{\mathcal{P}}} \rho \mathbf{K}_1 d\Sigma, \quad \mathfrak{M}(\mathcal{P}) \triangleq \iint_{\Sigma_{\mathcal{P}}} \rho \{(\boldsymbol{\rho} - \boldsymbol{\rho}_0) \times \mathbf{K}_1 + \mathbf{K}_2\} d\Sigma.$$

1. Balance of momentum. First Euler's law of motion of the shell.

The time rate of change of the momentum of an arbitrary shell part \mathcal{P} is equal to the total force acting on \mathcal{P} :

$$\frac{d}{dt} \mathfrak{P}(\mathcal{P}) = \mathfrak{F}, \quad \mathfrak{F} \triangleq \iint_{\Sigma_{\mathcal{P}}} \mathbf{f} d\Sigma + \int_{\omega_{\mathcal{P}}} \mathbf{t} d\omega. \quad (32)$$

Euler's motion laws

2. Balance of moment of momentum. Second Euler's law of motion of the shell. *The time rate of change of the moment of momentum of an arbitrary shell part \mathcal{P} about a fixed point ρ_0 is equal to the total moment about ρ_0 acting on \mathcal{P} :*

$$\frac{d}{dt} \mathfrak{M}(\mathcal{P}) = \mathfrak{C}, \quad (33)$$
$$\mathfrak{C} \triangleq \iint_{\Sigma_{\mathcal{P}}} \{(\boldsymbol{\rho} - \boldsymbol{\rho}_0) \times \mathbf{f} + \mathbf{m}\} d\Sigma + \int_{\omega_{\mathcal{P}}} \{(\boldsymbol{\rho} - \boldsymbol{\rho}_0) \times \mathbf{t} + \boldsymbol{\mu}\} d\omega.$$

Lagrange type principle

Let us assume that the external forces and couples are conservative.
In the Lagrange-type variational principle

$$\delta \mathcal{E}_1 = 0$$

we use the total energy functional

$$\mathcal{E}_1[\mathbf{r}, \mathbf{Q}] = \iint_{\Sigma} W d\Sigma - \mathcal{A}[\mathbf{r}, \mathbf{Q}], \quad (34)$$

where \mathcal{A} is the potential of the external loads.

Hu-Washizu type principle




For this principle the functional is given by

$$\begin{aligned}\mathcal{E}_2[\mathbf{r}, \mathbf{Q}, \mathbf{E}, \mathbf{K}, \mathbf{D}, \mathbf{P}_2] &= \iint_{\Sigma} \left\{ W(\mathbf{E}, \mathbf{K}) - \mathbf{D} \bullet (\mathbf{E} \cdot \mathbf{Q} - \mathbf{r}) \right. \\ &\quad \left. - \mathbf{P}_2 \bullet \left[\mathbf{K} - \frac{1}{2} \mathbf{r}^\alpha \otimes \left(\frac{\partial \mathbf{Q}}{\partial q^\alpha} \cdot \mathbf{Q}^T \right)_{\times} \right] \right\} d\Sigma \\ &\quad - \int_{\Omega_1} \boldsymbol{\nu} \cdot \mathbf{D} \cdot (\mathbf{r} - \boldsymbol{\rho}) d s - \mathcal{A}[\mathbf{r}, \mathbf{Q}].\end{aligned}$$

From the condition $\delta\mathcal{E}_2 = 0$ the equilibrium equations (24) and (25), the constitutive equations, and the relations (20) can be deduced. For this principle the natural boundary conditions are given by the relations (26)₁, (26)₂ and (26)₄, respectively.





Discussion (I)

One advantage of the theory is the correct description of kinematics of branching shells.

-  J. Chróścielewski, J. Makowski, and W. Pietraszkiewicz. *Statics and Dynamics of Multifolded Shells. Nonlinear Theory and Finite Element Method*. IPPT PAN, Warsaw, 2004.
-  V. Konopińska and W. Pietraszkiewicz. Exact resultant equilibrium conditions in the non-linear theory of branching and self-intersecting shells. *Int. J. Solids Struct.*, 44(1):352–369, 2007.
-  W. Pietraszkiewicz and V. Konopińska. On unique kinematics for the branching shells. *Int. J. Solids Struct.*, 48(14):2238–2244, 2011.





Discussion (II)

Several problem's solutions:

-  J. Chróścielewski and W. Witkowski. On some constitutive equations for micropolar plates. *ZAMM*, 90(1):53–64, 2010.
-  J. Chróścielewski and W. Witkowski. FEM analysis of Cosserat plates and shells based on some constitutive relations. *ZAMM*, 91(5):400–412, 2011.
-  J. Chróścielewski, W. Pietraszkiewicz, and W. Witkowski. On shear correction factors in the non-linear theory of elastic shells. *Int. J. Solids Struct.*, 47(25–26):3537–3545, 2010.
-  J. Chróścielewski, I. Kreja, A. Sabik, and W. Witkowski. Modeling of composite shells in 6-parameter nonlinear theory with drilling degree of freedom. *Mech. Adv. Materials Struct.*, 18(6):403–419, 2011.

Discussion (III)

Mathematical studies of the boundary-value problems:

-  M. Bîrsan and P. Neff. Existence theorems in the geometrically non-linear 6-parameter theory of elastic plates. *J. Elast.*, doi 10.1007/s10659-012-9405-2, 2012.
-  M. Bîrsan and P. Neff. Existence of minimizers in the geometrically non-linear 6-parameter resultant shell theory with drilling rotations. *Math. Mech. Solids*, doi 10.1177/1081286512466659, 2013.
-  M. Bîrsan and P. Neff. On the characterization of drilling rotation in the 6-parameter resultant shell theory. *arXiv preprint*, arXiv:1303.1979:1-9, 2013.
-  V.A. Eremeyev and L.P. Lebedev. Existence theorems in the linear theory of micropolar shells. *ZAMM*, 91(6):468-476, 2011.

Relations with the Nonlinear Elasticity

Equations (24) are the exact consequence of the 3D equilibrium equations of nonlinear elasticity. Applying the through-the-thickness integration procedure one obtains that \mathbf{D} and \mathbf{H} are given by the following formulae:

$$\mathbf{T} = \int_{-h/2}^{h/2} \mathbf{G} \cdot \mathbf{P} d\zeta, \quad \mathbf{M} = - \int_{-h/2}^{h/2} \mathbf{G} \cdot \mathbf{P} \times \mathbf{z} d\zeta, \quad (35)$$

where \mathbf{P} is the first Piola–Kirchhoff stress tensor, ζ is the coordinate normal to the base surface of the shell, \mathbf{z} is the base reference deviation and

$$\mathbf{G} \equiv -\mathbf{N} \times (\mathbf{A} - \zeta \mathbf{B}) \times \mathbf{N}$$

with $\mathbf{B} = -\nabla_{\zeta} \mathbf{N}$. The surface forces and couples \mathbf{f} and \mathbf{m} in (24) are also determined by the through-the-thickness integration procedure. Within the through-the-thickness integration one obtains exact consequence of the 3D equations of the nonlinear elasticity.

Small Strains

The following linearization is valid¹⁰. The primary kinematical variables are now the surface vectorial fields of translations and rotations. For small deformations we have

$$\boldsymbol{\rho} = \mathbf{P} + \mathbf{u}, \quad \mathbf{Q} \approx \mathbf{I} - \mathbf{I} \times \boldsymbol{\vartheta}. \quad (36)$$

where \mathbf{u} and $\boldsymbol{\vartheta}$ are the translations and microrotation vectors, respectively. These vectors satisfy the following restrictions

$$|\mathbf{u}| \ll 1, \quad |\boldsymbol{\vartheta}| \ll 1, \quad \|\nabla_s \mathbf{u}\| \ll 1, \quad \|\nabla_s \boldsymbol{\vartheta}\| \ll 1.$$

The surface strain tensors take the form

$$\mathbf{E} \approx \mathbf{I} + \boldsymbol{\epsilon}, \quad \mathbf{K} \approx \boldsymbol{\kappa}, \quad (37)$$

where the following linear strain measures are introduced:

$$\boldsymbol{\epsilon} = \nabla_s \mathbf{u} + \mathbf{A} \times \boldsymbol{\vartheta}, \quad \boldsymbol{\kappa} = \nabla_s \boldsymbol{\vartheta}.$$

¹⁰V. A. Eremeyev and L. M. Zubov. *Mechanics of Elastic Shells (in Russ.)*. Nauka, Moscow, 2008.

Strain Energy

Then the strain energy takes the form $W = W(\boldsymbol{\epsilon}, \boldsymbol{\kappa})$ and it follows

$$\mathbf{T} = \frac{\partial W}{\partial \boldsymbol{\epsilon}}, \quad \mathbf{M} = \frac{\partial W}{\partial \boldsymbol{\kappa}}. \quad (38)$$

The equilibrium equations in the linear theory are

$$\nabla_S \cdot \mathbf{T} + \mathbf{f} = \mathbf{0}, \quad \nabla_S \cdot \mathbf{M} + \mathbf{T}_\times + \mathbf{m} = \mathbf{0}, \quad (39)$$

and the boundary conditions transform to

$$\begin{aligned} \text{on } \omega_1 : \quad & \mathbf{u} = \mathbf{u}_0(s), \\ \text{on } \omega_2 : \quad & \boldsymbol{\nu} \cdot \mathbf{T} = \mathbf{t}(s), \\ \text{on } \omega_3 : \quad & \boldsymbol{\vartheta} = \boldsymbol{\vartheta}_0(s), \\ \text{on } \omega_4 : \quad & \boldsymbol{\nu} \cdot \mathbf{M} = \boldsymbol{\mu}(s), \end{aligned} \quad (40)$$

where $\mathbf{u}_0(s)$ and $\boldsymbol{\vartheta}_0(s)$ are given functions of s . These conditions define the translations and rotations on contour parts ω_1 and ω_3 , respectively.

Surface Strain Energy Density (Isotropic Material)

Quadratic approximation

$$\begin{aligned} 2W = & \alpha_1 \text{tr}^2 \mathbf{E}_{\parallel} + \alpha_3 \text{tr} \left(\mathbf{E}_{\parallel} \cdot \mathbf{E}_{\parallel}^T \right) \\ & + \alpha_4 \mathbf{N} \cdot \mathbf{E}^T \cdot \mathbf{E} \cdot \mathbf{N} + \beta_1 \text{tr}^2 \mathbf{K}_{\parallel} \\ & + \beta_3 \text{tr} \left(\mathbf{K}_{\parallel} \cdot \mathbf{K}_{\parallel}^T \right) + \beta_4 \mathbf{N} \cdot \mathbf{K}^T \cdot \mathbf{K} \cdot \mathbf{N}. \end{aligned}$$

α_k, β_k ($k = 1, 3, 4$) (elastic parameters - generalized shell stiffness parameters). The following abbreviations are used

$$\mathbf{E}_{\parallel} = \mathbf{E} \cdot \mathbf{A}, \quad \mathbf{K}_{\parallel} = \mathbf{K} \cdot \mathbf{A}.$$

Constitutive equations of anisotropic shells are discussed in the literature¹¹.

¹¹V. A. Eremeyev and W. Pietraszkiewicz. Local symmetry group in the general theory of elastic shells. *J. Elast.*, 85(2):125–152, 2006.

Simplest Case of Thin Shells

$$\alpha_1 = C\nu, \quad \alpha_3 = C(1 - \nu), \quad \alpha_4 = \alpha_s C(1 - \nu),$$
$$\beta_1 = D\nu, \quad \beta_3 = D(1 - \nu), \quad \beta_4 = \alpha_t D(1 - \nu)$$

$$\Rightarrow \quad C = \frac{Eh}{1 - \nu^2}, \quad D = \frac{Eh^3}{12(1 - \nu^2)}.$$

E, ν - Young's modulus and Poisson's ratio of the bulk material, h - shell thickness. α_s is an analogy to the shear correction factor introduced by Reissner ($\alpha_s = 5/6$) or Mindlin ($\alpha_s = \pi^2/12$). There are other suggestion^{12,13,14}.

¹²J. W. Hutchinson. Shear coefficients for Timoshenko beam theory. *Trans. ASME. J. Appl. Mech.*, 68(1):87-92, 2000

¹³F. Gruttmann and W. Wagner. Shear correction factors in Timoshenko's beam theory for arbitrary shaped cross-sections. *Comput. Mech.*, 27:199-207, 2001

¹⁴M. B. Rubin. On the quest for the best Timoshenko shear coefficient. *Trans. ASME. J. Appl. Mech.*, 70(1):154-157, 2003

Thank you for your attention!!!

Further questions:

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