Rayleigh variational principle and vibrations of prestressed shells

Victor A. Eremeyev

Rzeszow University of Technology, Rzeszów, Poland

Università degli Studi di Cagliari, Cagliari, June, 2017

Outline

1. Introduction
2. Basic equations of nonlinear shells
3. Linearized boundary-value problem
4. Free vibration of prestresses shell
5. Rayleigh Principle
6. Examples
7. Courant's principle: higher eigenfrequencies
8. Conclusions

General theory of shells

Here we use the general theory of shells presented e.g. in[1,2,3]

- The kinematics of the shell is determined by two kinematically independent fields of translations and rotations. 6 degrees of freedom: each point of the micropolar shell base surface has six degrees of freedom as in the rigid body dynamics.
- At the shell boundary acts forces and moments only.
- The drilling moment is also taken in account.
- Using certain constraints one can reduce the micropolar shell theory to the Kirchhoff–Love or Reissner–Mindlin shell models.

---

[1] A. Libai, J. G. Simmonds, The Nonlinear Theory of Elastic Shells, 2nd Edition, Cambridge University Press, Cambridge, 1998.

[2] J. Chróścielewski, J. Makowski, W. Pietraszkiewicz, Statics and Dynamics of Multifolded Shells. Nonlinear Theory and Finite Elelement Method (in Polish), Wydawnictwo IPPT PAN, Warszawa, 2004.

[3] V. A. Eremeyev, L. P. Lebedev, H. Altenbach, Foundations of Micropolar Mechanics, Springer, Heidelberg, 2012.

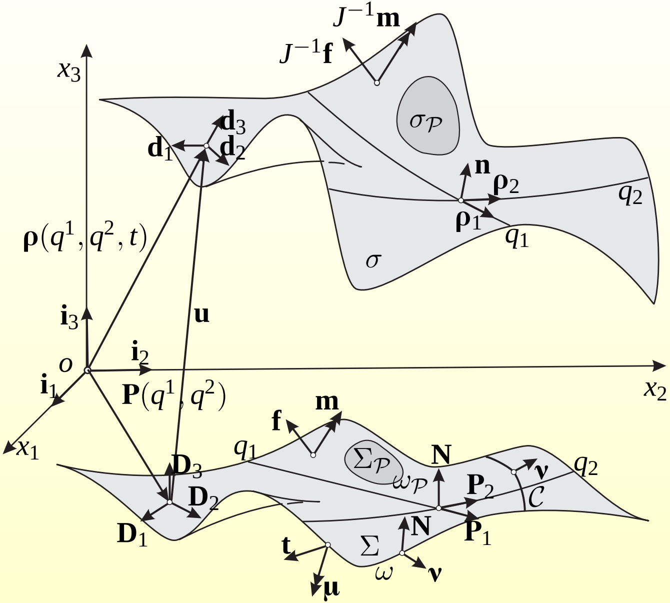
Shell Kinematics

- The deformation of the shell is described by mapping from one state called the *reference configuration* to another one called the *actual configuration*.
- In the reference configuration $\varkappa$: Σ is a base surface with position vector $\mathbf{P}(q^1, q^2)$ and with directors $\mathbf{D}_k(q^1, q^2,)$, $k = 1, 2, 3$.
- In the actual configuration χ: σ is the base surface with position vector $\boldsymbol{\rho}(q^1, q^2, t)$ and directors $\mathbf{d}_k(q^1, q^2, t)$, $k = 1, 2, 3$.

Hence, the shell is described by two kinematically independent fields

$$\boldsymbol{\rho} = \boldsymbol{\rho}(q^1, q^2, t) \quad \text{and} \quad \mathbf{Q} \equiv \mathbf{d}_k \otimes \mathbf{D}_k = \mathbf{Q}(q^1, q^2, t). \tag{1}$$

Shell Kinematics



Constitutive Equations of Elastic Shells (1)

According to the local action principle [4], the strain energy density W takes the form

$$W = W(\boldsymbol{\rho}, \nabla_{\varkappa}\boldsymbol{\rho}, \mathbf{Q}, \nabla_{\varkappa}\mathbf{Q}),$$

where

$$\nabla_{\varkappa} \triangleq \mathbf{P}^{\alpha} \frac{\partial}{\partial q^{\alpha}} \quad (\alpha, \beta = 1, 2),$$

$$\mathbf{P}^{\alpha} \cdot \mathbf{P}_{\beta} = \delta^{\alpha}_{\beta}, \quad \mathbf{P}^{\alpha} \cdot \mathbf{N} = 0, \quad \mathbf{P}_{\beta} = \frac{\partial \mathbf{P}}{\partial q^{\beta}}.$$

Here vectors $\mathbf{P}_{\beta}$ and $\mathbf{P}^{\alpha}$ denote the natural and reciprocal bases on Σ, respectively, $\mathbf{N}$ is the unit normal to Σ, δ^{α}_{β} is the Kronecker symbol, and $\nabla_{\varkappa}$ is the surface nabla operator on Σ.

---

[4] C. Truesdell, W. Noll, The nonlinear field theories of mechanics, in: S. Flügge (Ed.), Handbuch der Physik, Vol. III/3, Springer, Berlin, 1965, pp. 1–602.

Constitutive Equations of Elastic Shells (2)

From the principle of material frame-indifference [5] it follows that W depends on two surface strain measures $\mathbf{E}$ and $\mathbf{K}$:

$$W = W(\mathbf{E}, \mathbf{K}), \tag{2}$$

where

$$\mathbf{E} = \mathbf{F} \cdot \mathbf{Q}^T - \mathbf{A}, \quad \mathbf{K} = \frac{1}{2}\mathbf{P}^\alpha \otimes \left(\frac{\partial \mathbf{Q}}{\partial q^\alpha} \cdot \mathbf{Q}^T \right)_\times . \tag{3}$$

Here $\mathbf{F} = \nabla_\varkappa \boldsymbol{\rho}$, and $\mathbf{A} \triangleq \mathbf{I} - \mathbf{N} \otimes \mathbf{N}$, $\mathbf{I}$ is the unit 3D tensor,

$$\mathbf{T}_\times = (T^{mn}\mathbf{i}_m \otimes \mathbf{i}_n)_\times = T^{mn}\mathbf{i}_m \times \mathbf{i}_n$$

for any base $\mathbf{i}_m$, $\times$ denotes the vector (cross) product.

---

[5] C. Truesdell, W. Noll, The nonlinear field theories of mechanics, in: S. Flügge (Ed.), Handbuch der Physik, Vol. III/3, Springer, Berlin, 1965, pp. 1–602.

Vectorial Parameterizations of Strain Measures

Introducing

- the translation vector $\mathbf{u} = \boldsymbol{\rho} - \mathbf{P}$ and
- the finite rotation vector $\boldsymbol{\theta} = 2\mathbf{e}\tan\varphi/2$

we can express $\mathbf{Q}$, $\mathbf{E}$ and $\mathbf{K}$ as follows (see [6] for details)

$$\mathbf{Q} = \frac{1}{(4+\theta^2)}\left[(4-\theta^2)\mathbf{I} + 2\boldsymbol{\theta}\otimes\boldsymbol{\theta} - 4\mathbf{I}\times\boldsymbol{\theta}\right], \tag{4}$$

$$\mathbf{E} = (\mathbf{A} + \nabla_{\varkappa}\mathbf{u})\cdot\mathbf{Q}^T - \mathbf{A}, \qquad \theta^2 = \boldsymbol{\theta}\cdot\boldsymbol{\theta}, \tag{5}$$

$$\mathbf{K} = \frac{4}{4+\theta^2}\nabla_{\varkappa}\boldsymbol{\theta}\cdot\left(\mathbf{I} + \frac{1}{2}\mathbf{I}\times\boldsymbol{\theta}\right). \tag{6}$$

$\mathbf{Q}$ describes the rotation about axis with the unit vector $\mathbf{e}$ trough an angle φ.

---

[6] W. Pietraszkiewicz, V. A. Eremeyev, On vectorially parameterized natural strain measures of the non-linear Cosserat continuum, International Journal of Solids and Structures 46 (11–12) (2009) 2477–2480.

Lagrangian Equations of Motion

The Lagrangian equations of motion of the micropolar shell are

$$\nabla_{\varkappa} \cdot \mathbf{T}_{\varkappa} + \mathbf{f} = \rho_{\varkappa} \frac{\mathrm{d}\,\mathbf{K}_1}{\mathrm{d}\,t}, \tag{7}$$

$$\nabla_{\varkappa} \cdot \mathbf{M}_{\varkappa} + \left[\mathbf{F}^T \cdot \mathbf{T}_{\varkappa}\right]_{\times} + \mathbf{m} = \rho_{\varkappa} \left(\frac{\mathrm{d}\,\mathbf{K}_2}{\mathrm{d}\,t} + \mathbf{v} \times \boldsymbol{\Theta}_1^T \cdot \boldsymbol{\omega} \right), \tag{8}$$

where

$$\mathbf{T}_{\varkappa} = \mathbf{S}_1 \cdot \mathbf{Q}, \quad \mathbf{M}_{\varkappa} = \mathbf{S}_2 \cdot \mathbf{Q}, \tag{9}$$

$$\mathbf{S}_1 = \frac{\partial W}{\partial \mathbf{E}}, \quad \mathbf{S}_2 = \frac{\partial W}{\partial \mathbf{K}}, \tag{10}$$

$$\mathbf{K}_1 = \frac{\partial K}{\partial \mathbf{v}} = \mathbf{v} + \boldsymbol{\Theta}_1^T \cdot \boldsymbol{\omega}, \quad \mathbf{K}_2 = \frac{\partial K}{\partial \boldsymbol{\omega}} = \boldsymbol{\Theta}_1 \cdot \mathbf{v} + \boldsymbol{\Theta}_2 \cdot \boldsymbol{\omega}, \tag{11}$$

$$K(\mathbf{v}, \boldsymbol{\omega}) = \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \boldsymbol{\omega} \cdot \boldsymbol{\Theta}_1 \cdot \mathbf{v} + \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\Theta}_2 \cdot \boldsymbol{\omega}, \tag{12}$$

$$\mathbf{v} = \frac{\mathrm{d}\,\boldsymbol{\rho}}{\mathrm{d}\,t}, \qquad \boldsymbol{\omega} = \frac{1}{2} \left(\mathbf{Q}^T \cdot \frac{\mathrm{d}\,\mathbf{Q}}{\mathrm{d}\,t} \right)_{\times}.$$

Eulerian Equations of Motion

The Eulerian equations of motion of the micropolar shell are

$$\nabla_\chi \cdot \mathbf{T} + J^{-1}\mathbf{f} = \rho \frac{\mathrm{d}\,\mathbf{K}_1}{\mathrm{d}\,t}, \tag{13}$$

$$\nabla_\chi \cdot \mathbf{M} + \mathbf{T}_\times + J^{-1}\mathbf{m} = \rho \left(\frac{\mathrm{d}\,\mathbf{K}_2}{\mathrm{d}\,t} + \mathbf{v} \times \boldsymbol{\Theta}_1^T \cdot \boldsymbol{\omega} \right), \tag{14}$$

where

$$\nabla_\chi = \boldsymbol{\rho}^\alpha \frac{\partial}{\partial q^\alpha}, \quad \boldsymbol{\rho}^\alpha \cdot \boldsymbol{\rho}_\beta = \delta_\beta^\alpha, \;\; \boldsymbol{\rho}^\alpha \cdot \mathbf{n} = 0, \quad \boldsymbol{\rho}_\beta = \frac{\partial \boldsymbol{\rho}}{\partial q^\beta},$$

$$\mathbf{T} = J^{-1}\mathbf{F}^T \cdot \mathbf{T}_\varkappa, \quad \mathbf{M} = J^{-1}\mathbf{F}^T \cdot \mathbf{M}_\varkappa, \tag{15}$$

$$J = J(\mathbf{F}) = \sqrt{\frac{1}{2}\left\{ \left[\mathrm{tr}\,\left(\mathbf{F}\cdot\mathbf{F}^T\right)\right]^2 - \mathrm{tr}\,\left[\left(\mathbf{F}\cdot\mathbf{F}^T\right)^2\right] \right\}}.$$

Here $\mathbf{T}$ and $\mathbf{M}$ are Cauchy-type surface stress and couple stress tensors, ρ is the surface mass density in the actual configuration, ∇_χ is the surface nabla operator on σ related with $\nabla_\varkappa$ by the formula $\nabla_\varkappa = \mathbf{F} \cdot \nabla_\chi$, and $\mathbf{n}$ is the unit normal to σ.

Initial Boundary-Value Problem

Simplifications

$$\boldsymbol{\Theta}_1 = \mathbf{0}, \quad \boldsymbol{\Theta}_2 = \gamma \mathbf{I}, \tag{16}$$

where γ is a scalar measure of the rotatory inertia. The equations of motion (7) and (8) take more simple form

$$\nabla_\varkappa \cdot \mathbf{T}_\varkappa + \mathbf{f} = \rho_\varkappa \frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t}, \tag{17}$$

$$\nabla_\varkappa \cdot \mathbf{M}_\varkappa + \left[\mathbf{F}^T \cdot \mathbf{T}_\varkappa\right]_\times + \mathbf{m} = \rho_\varkappa \gamma \frac{\mathrm{d}\,\boldsymbol{\omega}}{\mathrm{d}\,t}. \tag{18}$$

Equations of motion are supplemented by the boundary conditions

$$\begin{array}{llll} \text{on } \omega_1: & \boldsymbol{\rho} = \mathbf{r}_0(s), & \text{on } \omega_2: & \boldsymbol{\nu} \cdot \mathbf{T}_\varkappa = \mathbf{t}(s), \\ \text{on } \omega_3: & \mathbf{Q} = \mathbf{h}(s), & \text{on } \omega_4: & \boldsymbol{\nu} \cdot \mathbf{M}_\varkappa = \boldsymbol{\mu}(s), \end{array} \tag{19}$$

and initial conditions

$$\boldsymbol{\rho}\big|_{t=0} = \boldsymbol{\rho}^\circ, \quad \mathbf{v}\big|_{t=0} = \mathbf{v}^\circ, \quad \mathbf{Q}\big|_{t=0} = \mathbf{Q}^\circ, \quad \boldsymbol{\omega}\big|_{t=0} = \boldsymbol{\omega}^\circ. \tag{20}$$

with given initial values $\boldsymbol{\rho}^\circ$, $\mathbf{v}^\circ$, $\mathbf{Q}^\circ$, $\boldsymbol{\omega}^\circ$, and given functions $\mathbf{r}_0(s)$, $\mathbf{h}(s)$, $\mathbf{t}(s)$, and $\boldsymbol{\mu}(s)$.

Isotropic Shell

Physically linear isotropic shell

$$\begin{aligned} 2W &= \alpha_1 \mathrm{tr}^2 \mathbf{E}_{\|} + \alpha_2 \mathrm{tr}\,\mathbf{E}_{\|}^2 + \alpha_3 \mathrm{tr}\left(\mathbf{E}_{\|} \cdot \mathbf{E}_{\|}^T\right) + \alpha_4 \mathbf{N} \cdot \mathbf{E}^T \cdot \mathbf{E} \cdot \mathbf{N} \\ &+ \beta_1 \mathrm{tr}^2 \mathbf{K}_{\|} + \beta_2 \mathrm{tr}\,\mathbf{K}_{\|}^2 + \beta_3 \mathrm{tr}\left(\mathbf{K}_{\|} \cdot \mathbf{K}_{\|}^T\right) + \beta_4 \mathbf{N} \cdot \mathbf{K}^T \cdot \mathbf{K} \cdot \mathbf{N}, \end{aligned} \tag{21}$$

where $\mathbf{E}_{\|} \triangleq \mathbf{E} \cdot \mathbf{A}$, $\mathbf{K}_{\|} \triangleq \mathbf{K} \cdot \mathbf{A}$. Here $\mathbf{S}_1$ and $\mathbf{S}_2$ have the form

$$\mathbf{S}_1 = \alpha_1 (\mathrm{tr}\,\mathbf{E}_{\|}) \mathbf{A} + \alpha_2 \mathbf{E}_{\|}^T + \alpha_3 \mathbf{E}_{\|} + \alpha_4 (\mathbf{E} \cdot \mathbf{N}) \otimes \mathbf{N}, \tag{22}$$

$$\mathbf{S}_2 = \beta_1 (\mathrm{tr}\,\mathbf{K}_{\|}) \mathbf{A} + \beta_2 \mathbf{K}_{\|}^T + \beta_3 \mathbf{K}_{\|} + \beta_4 (\mathbf{K} \cdot \mathbf{N}) \otimes \mathbf{N}. \tag{23}$$

Introducing the fourth-order tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ by the formulae

$$\begin{gathered} \mathbf{C}_1 = \alpha_1 \mathbf{A} \otimes \mathbf{A} + \alpha_2 \mathbf{P}_\alpha \otimes \mathbf{A} \otimes \mathbf{P}^\alpha \\ + \alpha_3 \mathbf{P}_\alpha \otimes \mathbf{P}_\beta \otimes \mathbf{P}^\alpha \otimes \mathbf{P}^\beta + \alpha_4 \mathbf{P}_\alpha \otimes \mathbf{N} \otimes \mathbf{P}^\alpha \otimes \mathbf{N}, \\ \mathbf{C}_2 = \beta_1 \mathbf{A} \otimes \mathbf{A} + \beta_2 \mathbf{P}_\alpha \otimes \mathbf{A} \otimes \mathbf{P}^\alpha \\ + \beta_3 \mathbf{P}_\alpha \otimes \mathbf{P}_\beta \otimes \mathbf{P}^\alpha \otimes \mathbf{P}^\beta + \beta_4 \mathbf{P}_\alpha \otimes \mathbf{N} \otimes \mathbf{P}^\alpha \otimes \mathbf{N}, \end{gathered}$$

we re-write (22) and (23) in a compact form $\mathbf{S}_1 = \mathbf{C}_1 : \mathbf{E}, \quad \mathbf{S}_2 = \mathbf{C}_2 : \mathbf{K}$.

Superimposed Infinitesimal Deformations

Let $\boldsymbol{\rho}_0$ and $\mathbf{Q}_0$ are the known static solution.
Superimposed infinitesimal deformations are

$$\boldsymbol{\rho}_* = \boldsymbol{\rho}_0 + \delta\boldsymbol{\rho}, \quad \mathbf{Q}_* = \mathbf{Q}_0 + \delta\mathbf{Q},$$

Since $\mathbf{Q}$ is an orthogonal tensor, the tensor $\mathbf{Q}^T \cdot \delta\mathbf{Q}$ is

$$\mathbf{Q}^T \cdot \delta\mathbf{Q} = -\mathbf{I} \times \boldsymbol{\psi},$$

where $\boldsymbol{\psi}$ is the infinitesimal rotation vector expressed by

$$\boldsymbol{\psi} = \frac{4}{4+\theta^2}\left(\delta\boldsymbol{\theta} + \frac{1}{2}\boldsymbol{\theta} \times \delta\boldsymbol{\theta}\right).$$

The increments of the strain measures are given by the formulae

$$\delta\mathbf{E} = (\nabla_{\varkappa}\delta\boldsymbol{\rho}) \cdot \mathbf{Q}_0^T + \mathbf{F}_0 \cdot \delta\mathbf{Q}^T = \mathbf{F}_0 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{Q}_0^T, \tag{24}$$

$$\delta\mathbf{K} = (\nabla_{\varkappa}\boldsymbol{\psi}) \cdot \mathbf{Q}^T = \mathbf{F}_0 \cdot \boldsymbol{\varkappa} \cdot \mathbf{Q}_0^T, \tag{25}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varkappa}$ are the linear strain measures given by

$$\boldsymbol{\varepsilon} = \nabla_{\chi}\mathbf{w} + \mathbf{A} \times \boldsymbol{\psi}, \quad \boldsymbol{\varkappa} = \nabla_{\chi}\boldsymbol{\psi}, \tag{26}$$

Linearization

Lagrangian linearized equations of motion

$$\nabla_{\varkappa} \cdot \delta \mathbf{T}_{\varkappa} = \rho_{\varkappa} \frac{\mathrm{d}^2 \mathbf{w}}{\mathrm{d}\, t^2}, \tag{27}$$

$$\nabla_{\varkappa} \cdot \delta \mathbf{M}_{\varkappa} + \left[(\nabla_{\varkappa} \mathbf{w})^T \cdot \mathbf{T}_{\varkappa} + \mathbf{F}_0^T \cdot \delta \mathbf{T}_{\varkappa} \right]_{\times} = \rho_{\varkappa} \gamma \frac{\mathrm{d}^2 \boldsymbol{\psi}}{\mathrm{d}\, t^2}, \tag{28}$$

Linearized boundary conditions

$$\begin{aligned} &\text{on } \omega_1 : \quad \mathbf{w} = \mathbf{0}, \qquad &&\text{on } \omega_2 : \quad \boldsymbol{\nu} \cdot \delta \mathbf{T}_{\varkappa} = \mathbf{0}, \\ &\text{on } \omega_3 : \quad \boldsymbol{\psi} = \mathbf{0}, \qquad &&\text{on } \omega_4 : \quad \boldsymbol{\nu} \cdot \delta \mathbf{M}_{\varkappa} = \mathbf{0}. \end{aligned} \tag{29}$$

Linearized Constitutive Relations

$$\delta\mathbf{T}_\varkappa = \delta\mathbf{S}_1\cdot\mathbf{Q}_0 + \mathbf{S}_1\cdot\delta\mathbf{Q} = \delta\mathbf{S}_1\cdot\mathbf{Q}_0 - \mathbf{T}_\varkappa\times\boldsymbol{\psi}, \tag{30}$$

$$\delta\mathbf{M}_\varkappa = \delta\mathbf{S}_2\cdot\mathbf{Q}_0 + \mathbf{S}_2\cdot\delta\mathbf{Q} = \delta\mathbf{S}_2\cdot\mathbf{Q}_0 - \mathbf{M}_\varkappa\times\boldsymbol{\psi}, \tag{31}$$

$$\delta\mathbf{S}_1 = \frac{\partial W}{\partial\mathbf{E}\partial\mathbf{E}} : \delta\mathbf{E} + \frac{\partial W}{\partial\mathbf{E}\partial\mathbf{K}} : \delta\mathbf{K}, \tag{32}$$

$$\delta\mathbf{S}_2 = \frac{\partial W}{\partial\mathbf{K}\partial\mathbf{E}} : \delta\mathbf{E} + \frac{\partial W}{\partial\mathbf{K}\partial\mathbf{K}} : \delta\mathbf{K}. \tag{33}$$

For the physically linear shell we have

$$\delta\mathbf{S}_1 = \mathbf{C}_1 : \delta\mathbf{E} = \mathbf{D}_1 : \boldsymbol{\varepsilon}, \quad \delta\mathbf{S}_2 = \mathbf{C}_2 : \delta\mathbf{K} = \mathbf{D}_2 : \boldsymbol{\varkappa},$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ are the fourth-order tensors given by

$$\begin{aligned}\mathbf{D}_1 =& \alpha_1\mathbf{A}\otimes\mathbf{F}_0^T\cdot\mathbf{P}_\alpha\otimes\mathbf{Q}_0^T\cdot\mathbf{P}^\alpha + \alpha_2\mathbf{P}_\alpha\otimes\mathbf{P}_\beta\otimes\mathbf{F}_0^T\cdot\mathbf{P}^\beta\otimes\mathbf{Q}_0^T\cdot\mathbf{P}^\alpha\\ &+ \alpha_3\mathbf{P}_\alpha\otimes\mathbf{P}_\beta\otimes\mathbf{F}_0^T\cdot\mathbf{P}^\alpha\otimes\mathbf{Q}_0^T\cdot\mathbf{P}^\beta + \alpha_4\mathbf{P}_\alpha\otimes\mathbf{N}\otimes\mathbf{F}_0^T\cdot\mathbf{P}^\alpha\otimes\mathbf{Q}_0^T\cdot\mathbf{N},\\ \mathbf{D}_2 =& \beta_1\mathbf{A}\otimes\mathbf{F}_0^T\cdot\mathbf{P}_\alpha\otimes\mathbf{Q}_0^T\cdot\mathbf{P}^\alpha + \beta_2\mathbf{P}_\alpha\otimes\mathbf{P}_\beta\otimes\mathbf{F}_0^T\cdot\mathbf{P}^\beta\otimes\mathbf{Q}_0^T\cdot\mathbf{P}^\alpha\\ &+ \beta_3\mathbf{P}_\alpha\otimes\mathbf{P}_\beta\otimes\mathbf{F}_0^T\cdot\mathbf{P}^\alpha\otimes\mathbf{Q}_0^T\cdot\mathbf{P}^\beta + \beta_4\mathbf{P}_\alpha\otimes\mathbf{N}\otimes\mathbf{F}_0^T\cdot\mathbf{P}^\alpha\otimes\mathbf{Q}_0^T\cdot\mathbf{N}.\end{aligned}$$

Linearized Eulerian Equations of Motion and BCs

Introducing the tensors

$$\boldsymbol{\Phi}_1 = J_0^{-1}\mathbf{F}_0^T \cdot \delta\mathbf{T}_{\varkappa}, \quad \boldsymbol{\Phi}_2 = J_0^{-1}\mathbf{F}_0^T \cdot \delta\mathbf{M}_{\varkappa}, \tag{34}$$

where $J_0 = J(\mathbf{F}_0)$, we transform (27) and (28) into the linearized equation of motion in the actual configuration χ_0

$$\nabla_\chi \cdot \boldsymbol{\Phi}_1 = \rho \frac{\mathrm{d}^2\mathbf{w}}{\mathrm{d}t^2}, \tag{35}$$

$$\nabla_\chi \cdot \boldsymbol{\Phi}_2 + \left[(\nabla_\chi \mathbf{w})^T \cdot \mathbf{T} + \boldsymbol{\Phi}_1\right]_\times = \rho\gamma \frac{\mathrm{d}^2\boldsymbol{\psi}}{\mathrm{d}t^2}. \tag{36}$$

For the physically linear isotropic shell $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ are

$$\boldsymbol{\Phi}_1 = \mathbf{H}_1 : \boldsymbol{\varepsilon} - \mathbf{T} \times \boldsymbol{\psi}, \quad \boldsymbol{\Phi}_1 = \mathbf{H}_2 : \boldsymbol{\varkappa} - \mathbf{M} \times \boldsymbol{\psi},$$

$$\mathbf{H}_1 = J_0^{-1}\mathbf{F}_0^T \cdot \mathbf{D}_1, \quad \mathbf{H}_2 = J_0^{-1}\mathbf{F}_0^T \cdot \mathbf{D}_2.$$

The linearized Eulerian boundary conditions are

$$\begin{aligned} &\text{on } \ell_1: \quad \mathbf{w} = \mathbf{0}, && \text{on } \ell_2: \quad \boldsymbol{\eta} \cdot \boldsymbol{\Phi}_1 = \mathbf{0}, \\ &\text{on } \ell_3: \quad \boldsymbol{\psi} = \mathbf{0}, && \text{on } \ell_4: \quad \boldsymbol{\eta} \cdot \boldsymbol{\Phi}_2 = \mathbf{0}. \end{aligned} \tag{37}$$

Free Vibration of Prestresses Shell

$$\mathbf{w} = \mathbf{W}(q^1, q^2)e^{i\Omega t}, \quad \boldsymbol{\psi} = \boldsymbol{\Psi}(q^1, q^2)e^{i\Omega t}.$$

Substituting the latter relations into (35) and (37) we obtain the boundary-value problem for the physically linear isotropic prestressed micropolar shell

$$\nabla_\chi \cdot \boldsymbol{\Phi}_1 = -\rho\Omega^2\mathbf{W}, \tag{38}$$

$$\nabla_\chi \cdot \boldsymbol{\Phi}_2 + \left[(\nabla_\chi \mathbf{w})^T \cdot \mathbf{T} + \boldsymbol{\Phi}_1\right]_\times = -\rho\gamma\Omega^2\boldsymbol{\Psi}, \tag{39}$$

$$\begin{array}{llll} \text{on } \ell_1: & \mathbf{W} = \mathbf{0}, & \text{on } \ell_2: & \boldsymbol{\eta} \cdot \boldsymbol{\Phi}_1 = \mathbf{0}, \\ \text{on } \ell_3: & \boldsymbol{\Psi} = \mathbf{0}, & \text{on } \ell_4: & \boldsymbol{\eta} \cdot \boldsymbol{\Phi}_2 = \mathbf{0}, \end{array} \tag{40}$$

where

$$\boldsymbol{\Phi}_1 = \mathbf{H}_1 : \boldsymbol{\varepsilon} - \mathbf{T} \times \boldsymbol{\Psi}, \quad \boldsymbol{\Phi}_1 = \mathbf{H}_2 : \boldsymbol{\varkappa} - \mathbf{M} \times \boldsymbol{\Psi}, \tag{41}$$

$$\boldsymbol{\varepsilon} = \nabla_\chi \mathbf{W} + \mathbf{A} \times \boldsymbol{\Psi}, \quad \boldsymbol{\varkappa} = \nabla_\chi \boldsymbol{\Psi}.$$

Comparison Problem

Additionally we consider the linear boundary-value problem of the micropolar shell without initial deformation, that is when $\chi_0 = \varkappa$, which is given by

$$\nabla_\chi \cdot \mathbf{\Phi}_1^0 = -\rho\Omega^2 \mathbf{W}, \quad \nabla_\chi \cdot \mathbf{\Phi}_2^0 + \mathbf{\Phi}_{1\times}^0 = -\rho\gamma\Omega^2 \mathbf{\Psi}, \tag{42}$$

$$\begin{aligned} &\text{on } \ell_1: \quad \mathbf{W} = \mathbf{0}, \qquad &&\text{on } \ell_2: \quad \boldsymbol{\eta} \cdot \mathbf{\Phi}_1^0 = \mathbf{0}, \\ &\text{on } \ell_3: \quad \mathbf{\Psi} = \mathbf{0}, \qquad &&\text{on } \ell_4: \quad \boldsymbol{\eta} \cdot \mathbf{\Phi}_2^0 = \mathbf{0}, \end{aligned} \tag{43}$$

$$\mathbf{\Phi}_1^0 = \mathbf{C}_1 : \boldsymbol{\varepsilon}, \quad \mathbf{\Phi}_1^0 = \mathbf{C}_2 : \boldsymbol{\varkappa}. \tag{44}$$

Comparison of $\mathbf{\Phi}_1^0$ and $\mathbf{\Phi}_1$, $\mathbf{\Phi}_2^0$ and $\mathbf{\Phi}_2$ shows that difference between these boundary-value problems consists of

1. difference between the elastic moduli tensors $\mathbf{C}_\alpha$ and $\mathbf{H}_\alpha$, $\alpha = 1, 2$, and
2. existence of initial stress tensors $\mathbf{T}$ and $\mathbf{M}$ in $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$.

In what follows we show the influence on the eigen-frequencies of the prestressed shell using the variational approach.

Second Variation of the Total Energy

The total potential energy of the shell is

$$\Pi = \iint\limits_{\Sigma} W \, \mathrm{d}\Sigma - \iint\limits_{\Sigma} \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}\Sigma - \int\limits_{\omega_2} \mathbf{t} \cdot \mathbf{u} \, \mathrm{d}s.$$

The second variation of energy is

$$\delta^2 \Pi = 2 \iint\limits_{\sigma} w \, \mathrm{d}\sigma, \quad w = w_1 + w_2, \tag{45}$$

where

$$\begin{aligned} w_1(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) = &\quad \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{H}_1 : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varkappa} : \mathbf{H}_2 : \boldsymbol{\varkappa}, \\ w_2(\boldsymbol{\psi}, \boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) = &\quad \operatorname{tr}\left(\boldsymbol{\psi} \times \mathbf{T}^T \cdot \boldsymbol{\varepsilon}\right) - \frac{1}{2}\operatorname{tr}\left(\boldsymbol{\psi} \times \mathbf{T}^T \times \boldsymbol{\psi}\right) + \frac{1}{2}\operatorname{tr}\left(\boldsymbol{\psi} \times \mathbf{M}^T \cdot \boldsymbol{\varkappa}\right). \end{aligned} \tag{46}$$

If $\chi_0 = \varkappa$, that is $\mathbf{T} = \mathbf{M} = \mathbf{0}$, then

$$w = w_0 \equiv \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{C}_1 : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varkappa} : \mathbf{C}_2 : \boldsymbol{\varkappa}.$$

Rayleigh Principle

The modes of the shell eigen-oscillations are stationary points of the energy functional

$$\mathcal{E}[\mathbf{W},\boldsymbol{\Psi}] = \iint\limits_{\sigma} [w_1(\boldsymbol{\varepsilon},\boldsymbol{\varkappa}) + w_2(\boldsymbol{\Psi},\boldsymbol{\varepsilon},\boldsymbol{\varkappa})]\,\mathrm{d}\,\sigma, \tag{47}$$

where

$$\boldsymbol{\varepsilon} = \nabla_{\chi}\mathbf{W} + \mathbf{A}\times\boldsymbol{\Psi}, \qquad \boldsymbol{\varkappa} = \nabla_{\chi}\boldsymbol{\Psi},$$

on the set of functions that satisfy the kinematic boundary conditions

$$\text{on } \ell_1 : \quad \mathbf{W} = \mathbf{0} \qquad \text{and on } \ell_3 : \quad \boldsymbol{\Psi} = \mathbf{0} \tag{48}$$

and the restriction

$$\mathcal{K}(\mathbf{W},\boldsymbol{\Psi}) \equiv \frac{1}{2}\iint\limits_{\sigma} \rho\,(\mathbf{W}\cdot\mathbf{W} + \gamma\boldsymbol{\Psi}\cdot\boldsymbol{\Psi})\;\mathrm{d}\,\sigma = 1. \tag{49}$$

Here the functions $\mathbf{W}$, $\boldsymbol{\Psi}$ are the oscillation amplitudes for the translations and rotations, respectively.

Rayleigh quotient

The Rayleigh variational principle is equivalent to the stationary principle for the Rayleigh quotient

$$\mathcal{R}[\mathbf{W},\boldsymbol{\Psi}] = \frac{\mathcal{E}[\mathbf{W},\boldsymbol{\Psi}]}{\mathcal{K}(\mathbf{W},\boldsymbol{\Psi})}, \tag{50}$$

that is defined on kinematically admissible functions $\mathbf{W}$, $\boldsymbol{\Psi}$.
The proof is standard and mimics one which can be found for example in the textbook[7].
The Rayleigh quotient of the shell without initial stresses is

$$\mathcal{R}_0[\mathbf{W},\boldsymbol{\Psi}] = \frac{\mathcal{E}_0[\mathbf{W},\boldsymbol{\Psi}]}{\mathcal{K}(\mathbf{W},\boldsymbol{\Psi})}, \quad \mathcal{E}_0[\mathbf{W},\boldsymbol{\Psi}] = \iint\limits_{\sigma} w_0(\boldsymbol{\varepsilon},\boldsymbol{\varkappa})\,\mathrm{d}\,\sigma. \tag{51}$$

---

[7] V. L. Berdichevsky, Variational Principles of Continuum Mechanics. I. Fundamentals, Springer, Heidelberg, 2009.

Comparison of Eigen-Frequencies

Least squared eigenfrequencies are

$$\Omega^2_{\min} = \inf \mathcal{R}[\mathbf{W}, \mathbf{\Psi}], \quad {\Omega_0}^2_{\min} = \inf \mathcal{R}_0[\mathbf{W}, \mathbf{\Psi}].$$

By the Courant minimax principle[8], the Rayleigh quotient (50) allows us to estimate the values of higher eigen-frequencies. It is obvious that difference between $\mathcal{E}$ and $\mathcal{E}_0$ consist of two terms: difference in elastic moduli, that is the difference between $\mathbf{C}_1$ and $\mathbf{H}_1$, $\mathbf{C}_2$ and $\mathbf{H}_2$, and the term w_2 depending on initial stress and couple stress tensors.

Result. If $w(\mathbf{\Psi}, \boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) \le w_0(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa})$ then

$$\Omega_k \le \Omega_k^0, \quad k = 1, 2, \ldots$$

and vice versa.

---

[8] R. Courant, D. Hilbert, Methods of Mathematical Physics, Vol. 1, Wiley, New York, 1991.

Uniform stretching

Let us assume that $\mathbf{C}_1 \approx \mathbf{H}_1$, $\mathbf{C}_2 \approx \mathbf{H}_2$ and uniform stretching of the shell with $\mathbf{T} = p\mathbf{A}$, $\mathbf{M} = \mathbf{0}$, p is the uniform tension. Here we have

$$w - w_0 = w_2.$$

We have

$$\begin{aligned} w_2(\boldsymbol{\Psi}, \boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) =& p\,\mathrm{tr}\,(\boldsymbol{\Psi} \times \mathbf{A} \cdot \boldsymbol{\varepsilon}) - \frac{p}{2}\mathrm{tr}\,(\boldsymbol{\Psi} \times \mathbf{A} \times \boldsymbol{\Psi}) \\ =& p\,\mathrm{tr}\,(\boldsymbol{\Psi} \times \nabla_\chi \mathbf{W}) + \frac{p}{2}\mathrm{tr}\,(\boldsymbol{\Psi} \times \mathbf{A} \times \boldsymbol{\Psi}) \\ =& p\,\mathrm{tr}\,(\boldsymbol{\Psi} \times \nabla_\chi \mathbf{W}) + \frac{p}{2}[\boldsymbol{\Psi} \cdot \boldsymbol{\Psi} + (\boldsymbol{\Psi} \cdot \mathbf{N})^2]. \end{aligned}$$

Assuming $\nabla_\chi \mathbf{W} = \mathbf{0}$ we obtain

$$w_2 = p/2[\boldsymbol{\Psi} \cdot \boldsymbol{\Psi} + (\boldsymbol{\Psi} \cdot \mathbf{N})^2]$$

and the sign of w_2 coincides with the sign of p. This case is similar to the dependence of eigen-frequency of a string on its tension: stretching ($p > 0$) leads to increase while compression ($p < 0$) leads to decrease of eigen-frequencies in comparison with undeformable shell

Spherical shell under hydrostatic pressure

Here $\mathbf{f} = -q\mathbf{N}$, $\mathbf{m} = \mathbf{0}$. We introduce the Lagrangian and Eulerian spherical coordinates

$$X_1 = R\cos\Phi\cos\Theta, \quad X_2 = R\sin\Phi\cos\Theta, \quad X_3 = R\sin\Theta,$$
$$\mathbf{e}_R = (\mathbf{i}_1\cos\Phi + \mathbf{i}_2\sin\Phi)\cos\Theta + \mathbf{i}_3\sin\Theta,$$
$$\mathbf{e}_\Phi = -\mathbf{i}_1\sin\Phi + \mathbf{i}_2\cos\Phi,$$
$$\mathbf{e}_\Theta = -(\mathbf{i}_1\cos\Phi + \mathbf{i}_2\sin\Phi)\sin\Theta + \mathbf{i}_3\cos\Theta,$$
$$x_1 = r\cos\phi\cos\theta, \quad x_2 = r\sin\phi\cos\theta, \quad x_3 = r\sin\theta,$$
$$\mathbf{e}_r = (\mathbf{i}_1\cos\phi + \mathbf{i}_2\sin\phi)\cos\theta + \mathbf{i}_3\sin\theta,$$
$$\mathbf{e}_\phi = -\mathbf{i}_1\sin\phi + \mathbf{i}_2\cos\phi,$$
$$\mathbf{e}_\theta = -(\mathbf{i}_1\cos\phi + \mathbf{i}_2\sin\phi)\sin\theta + \mathbf{i}_3\cos\theta,$$

As a result we assume the deformation as

$$r = a, \quad \phi = \Phi, \quad \theta = \Theta. \tag{52}$$

$$\mathbf{Q} = \mathbf{I}. \tag{53}$$

Spherical shell under hydrostatic pressure (2)

Using the definition

$$\nabla_{\varkappa} = \frac{1}{a_0 \cos\Theta} \mathbf{e}_{\Phi} \frac{\partial}{\partial \Phi} + \frac{1}{a_0} \mathbf{e}_{\Theta} \frac{\partial}{\partial \Theta}$$

we obtain that

$$\mathbf{F} = \frac{a}{a_0}(\mathbf{e}_{\Phi} \otimes \mathbf{e}_{\phi} + \mathbf{e}_{\Theta} \otimes \mathbf{e}_{\theta}) = \lambda \mathbf{A}, \quad \lambda = \frac{a}{a_0}, \tag{54}$$

$$\mathbf{E} = (\lambda - 1)\mathbf{A}, \quad \mathbf{K} = \mathbf{0}, \tag{55}$$

and

$$\mathbf{T}_{\varkappa} = (2\alpha_1 + \alpha_2 + \alpha_3)(\lambda - 1)\mathbf{A}, \quad \mathbf{M}_{\varkappa} = \mathbf{0}. \tag{56}$$

Equilibrium conditions reduce to one algebraic equation $T = q$, where $T = (2\alpha_1 + \alpha_2 + \alpha_3)(\lambda - 1)$. We find that

$$\mathbf{H}_1 = \mathbf{C}_1, \qquad \mathbf{H}_2 = \mathbf{C}_2. \tag{57}$$

As a result we have $w_1 = w_0$ while

$$w_2(\boldsymbol{\psi}, \boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) = T \,\mathrm{tr}\, (\boldsymbol{\psi} \times \nabla_{\chi} \mathbf{W}) + \frac{T}{2}[\boldsymbol{\psi} \cdot \boldsymbol{\psi} + (\boldsymbol{\psi} \cdot \mathbf{N})^2].$$

Eversion of a spherical shell

The eversion of the spherical shell is described by

$$r = a, \quad \phi = \Phi, \quad \theta = -\Theta, \tag{58}$$

$$\mathbf{Q} = \mathbf{e}_\Phi \otimes \mathbf{e}_\phi - \mathbf{e}_R \otimes \mathbf{e}_r - \mathbf{e}_\Theta \otimes \mathbf{e}_\theta. \tag{59}$$

Here

$$\mathbf{F} = \lambda(\mathbf{e}_\Phi \otimes \mathbf{e}_\phi - \mathbf{e}_\Theta \otimes \mathbf{e}_\theta), \quad \lambda = \frac{a}{a_0}, \tag{60}$$

$$\mathbf{E} = (\lambda - 1)\mathbf{A}, \quad \mathbf{K} = \frac{2}{a_0}(\mathbf{e}_\Theta \otimes \mathbf{e}_\Phi - \mathbf{e}_\Phi \otimes \mathbf{e}_\Theta), \tag{61}$$

$$\text{and} \quad \mathbf{S}_1 = T\mathbf{A}, \quad \mathbf{S}_2 = (\beta_3 - \beta_2)\mathbf{K}. \tag{62}$$

Equilibrium conditions reduce to $T = 0$ while

$$\mathbf{M} = \frac{2}{a_0}(\beta_3 - \beta_1)(\mathbf{e}_\phi \otimes \mathbf{e}_\theta - \mathbf{e}_\theta \otimes \mathbf{e}_\phi) \neq \mathbf{0}.$$

The eversion of a spherical shell demonstrates an example of zero initial stresses with non-zero initial couple stresses ($\mathbf{f} = \mathbf{0} = \mathbf{m}$).

Eversion of a spherical shell (2)

For such initial stressed state we obtain again that

$$\mathbf{H}_1 = \mathbf{C}_1, \qquad \mathbf{H}_2 = \mathbf{C}_2$$

and

$$w_1 = w_0, \quad w_2(\boldsymbol{\psi}, \boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) = \frac{1}{2}\mathrm{tr}\,\left(\boldsymbol{\psi} \times \mathbf{M}^T \cdot \boldsymbol{\varkappa}\right). \tag{63}$$

Unlike to the case of the shell loaded by initial pressure for eversion there is not an external loading parameter.

Conclusion

In both considered cases we proved that the changes in eigen-frequencies spectrum due to initial stresses are determined by initial stress fields $\mathbf{T}$ and $\mathbf{M}$.

Courant's principle: preliminary notations

To formulate Courant's minimax principle, we introduce the space $\mathbf{H} = \{\mathbf{h} = (\mathbf{W}, \boldsymbol{\Psi})\}$ of functions $\mathbf{h}$ satisfying (48) with respect to the energy norm defined by

$$\|\mathbf{h}\|^2_{\mathbf{H}} = \frac{1}{2} \iint\limits_{\sigma} \left(\boldsymbol{\varepsilon} : \mathbf{D}_1 : \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} : \mathbf{D}_2 : \boldsymbol{\varkappa} + \boldsymbol{\varkappa} : \mathbf{D}_3 : \boldsymbol{\varkappa}\right) \mathrm{d}\sigma. \tag{64}$$

Let the eigenvalues for the problem under consideration be ordered as $0 < \Omega_{\text{min}} = \Omega_1 \leq \Omega_2 \leq \cdots$. To each Ω_k there corresponds a unique eigenmode $\mathbf{h}_k$.

Let $k > 1$, and denote by $\mathbf{H}^{(k)}$ the subspace of $\mathbf{H}$ spanned by $k-1$ arbitrarily chosen elements $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{k-1}$ of $\mathbf{H}$. The space $\mathbf{H}^{(k)}_{\perp}$ is its "orthogonal" complement in $\mathbf{H}$:

$$\mathbf{H}^{(k)}_{\perp} = \left\{\mathbf{h} \in \mathbf{H} \mid \langle \mathbf{h}, \mathbf{g}_1 \rangle_{\mathbf{L}} = \langle \mathbf{h}, \mathbf{g}_2 \rangle_{\mathbf{L}} = \cdots = \langle \mathbf{h}, \mathbf{g}_{k-1} \rangle_{\mathbf{L}} = 0\right\},$$

$$\langle \mathbf{g}, \mathbf{h} \rangle_{\mathbf{L}} \equiv \langle (\mathbf{W}_g, \boldsymbol{\Psi}_g), (\mathbf{W}_h, \boldsymbol{\Psi}_h) \rangle_{\mathbf{L}} = \frac{1}{2} \iint\limits_{\sigma} \rho \left(\mathbf{W}_h \cdot \mathbf{W}_g + \gamma \boldsymbol{\Psi}_h \cdot \boldsymbol{\Psi}_g\right) \, \mathrm{d}\,\sigma.$$

Courant's principle: higher eigenfrequencies

$\mathbf{H}_\perp^{(k)}$ is a closed subspace of Hilbert space $\mathbf{H}$ as the orthogonal complement to the finite dimensional space spanned by $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{k-1}$. By $\widehat{\mathbf{H}}_\perp^{(k)}$ we denote the subset of elements of $\mathbf{H}_\perp^{(k)}$ with the constraint $\langle \mathbf{h}, \mathbf{h} \rangle_\mathbf{L} = 1$, i.e., $\widehat{\mathbf{H}}_\perp^{(k)} = \{\mathbf{h} \in \mathbf{H}_\perp^{(k)} \mid \langle \mathbf{h}, \mathbf{h} \rangle_\mathbf{L} = 1\}$. Let us introduce $d[\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{k-1}] = \inf\limits_{\widehat{\mathbf{H}}_\perp^{(k)}} \mathcal{R}(\mathbf{h})$.

Courant's minimax principle

states that by taking the suprema of the quantities $d[\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{k-1}]$ over all possible combinations $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{k-1}$ in $\mathbf{H}$, we obtain the eigenfrequencies

$$\Omega_k^2 = \sup_{\mathbf{g}_1, \ldots, \mathbf{g}_{k-1}} d[\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{k-1}] = \sup_{\mathbf{g}_1, \ldots, \mathbf{g}_{k-1}} \inf_{\widehat{\mathbf{H}}_\perp^{(k)}} \mathcal{R}(\mathbf{h}). \qquad (65)$$

These maximum-minimum values are attained if the elements $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{k-1}$ coincide with the first $k-1$ eigenmodes.

General remarks 1.

If

$$\mathcal{R}_1 \geq \mathcal{R}_2 \quad \text{for all } \mathbf{h} \in \mathbf{H} \tag{66}$$

then

$$\Omega_{1k} \geq \Omega_{2k} \qquad (k = 1, 2, \ldots). \tag{67}$$

Indeed, it follows that for any set $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{k-1}$ we have

$$\inf_{\widehat{\mathbf{H}}_{\perp}^{(k)}} \mathcal{R}_1(\mathbf{h}) \geq \inf_{\widehat{\mathbf{H}}_{\perp}^{(k)}} \mathcal{R}_2(\mathbf{h}).$$

Hence

$$\Omega_{1k}^2 = \sup_{\mathbf{g}_1,\ldots,\mathbf{g}_{k-1}} \inf_{\widehat{\mathbf{H}}_{\perp}^{(k)}} \mathcal{R}_1(\mathbf{h}) \geq \sup_{\mathbf{g}_1,\ldots,\mathbf{g}_{k-1}} \inf_{\widehat{\mathbf{H}}_{\perp}^{(k)}} \mathcal{R}_2(\mathbf{h}) = \Omega_{2k}^2.$$

General remarks 2.

Let $\mathbf{H}$ be the space over which we minimize Rayleigh's quotient for a shell, and let $\mathbf{H}_1$ be the corresponding space for the same shell subjected to a constraint of geometrical nature. Then

$$\mathbf{H}_1 \subset \mathbf{H}.$$

Here we also can demonstrate that the least eigenfrequencies satisfy $\Omega_{1\,\min} \geq \Omega_{\min}$. Using Courant's minimax principle, we can prove that

$$\Omega_{1k} \geq \Omega_k$$

for $k = 1, 2, \ldots$.

Dependence on boundary conditions

Let us consider three boundary-value problems, all with a portion ℓ_1 of the shell boundary clamped.

1. Portion ℓ_2 of edge free:

$$\mathbf{W}\big|_{\ell_1} = \mathbf{0}, \quad \boldsymbol{\Psi}\big|_{\ell_1} = \mathbf{0}, \quad \boldsymbol{\eta} \cdot \boldsymbol{\Phi}_1\big|_{\ell_2} = \mathbf{0}, \quad \boldsymbol{\eta} \cdot \boldsymbol{\Phi}_2\big|_{\ell_2} = \mathbf{0},$$

2. Portion ℓ_2 of edge simply supported:

$$\mathbf{W}\big|_{\ell} = \mathbf{0}, \quad \boldsymbol{\Psi}\big|_{\ell_1} = \mathbf{0}, \quad \boldsymbol{\eta} \cdot \boldsymbol{\Phi}_2\big|_{\ell_2} = \mathbf{0}.$$

3. Entire edge ℓ clamped:

$$\mathbf{W}\big|_{\ell} = \mathbf{0}, \quad \boldsymbol{\Psi}\big|_{\ell} = \mathbf{0}.$$

By Courant’s principle, one can easily establish that

$$\Omega_k^{(3)} \geq \Omega_k^{(2)} \geq \Omega_k^{(1)} \qquad (k = 1, 2, \ldots)$$

where $\Omega_k^{(i)}$ is the kth eigenfrequency for the ith problem.

Dependence on rotational inertia

If $K_1 \geq K_2$, then the corresponding eigenfrequencies satisfy $\Omega_k^{(1)} \leq \Omega_k^{(2)}$ for $k = 1, 2, \ldots$.

Let us analyze how the rotational inertia affects the shell eigenfrequencies. Instead of (12), we use the following form of the kinetic energy:

$$K = \frac{1}{2}\mathbf{v} \cdot \mathbf{v} + \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\Theta} \cdot \boldsymbol{\omega},$$

where $\boldsymbol{\Theta}$ is a positive definite second-order tensor of rotational inertia and $\theta_1 \leq \theta_2 \leq \theta_3$, we get

$$\theta_1 \boldsymbol{\omega} \cdot \boldsymbol{\omega} \leq \boldsymbol{\omega} \cdot \boldsymbol{\Theta} \cdot \boldsymbol{\omega} \leq \theta_3 \boldsymbol{\omega} \cdot \boldsymbol{\omega}.$$

Then the eigenfrequencies for a shell where the kinetic energy includes the inertia tensor $\boldsymbol{\Theta}$ are bounded below by the eigenfrequencies calculated for shells with scalar inertia measures $\tilde{\gamma} = \theta_1$ and $\hat{\gamma} = \theta_3$ respectively:

$$\widehat{\Omega}_k \leq \Omega_k \leq \widetilde{\Omega}_k \qquad (k = 1, 2, \ldots).$$

Boundary approximations

Let us consider a plate with a clamped boundary ℓ and two approximated polygonal contours ℓ_n and ℓ_m, where ℓ_n is an inscribed polygon, ℓ_m is a circumscribed polygon, and n and m are the numbers of sides of ℓ_n and ℓ_m, respectively (Fig. 1).

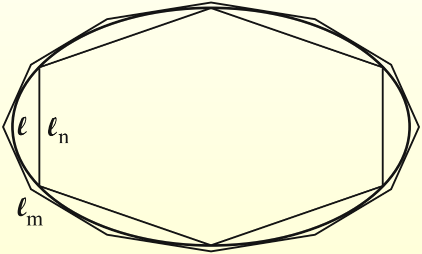


Figure: Elliptic plate contour ℓ, inscribed polygon contour ℓ_n and circumscribed polygon contour ℓ_m for $n = 6$, $m = 12$

Boundary approximations

We consider three problems with the respective boundary conditions

$$\mathbf{W}\Big|_{\ell} = \mathbf{0}, \quad \Psi\Big|_{\ell} = \mathbf{0}, \quad \mathbf{W}\Big|_{\ell_n} = \mathbf{0}, \quad \Psi\Big|_{\ell_n} = \mathbf{0}, \quad \mathbf{W}\Big|_{\ell_m} = \mathbf{0}, \quad \Psi\Big|_{\ell_m} = \mathbf{0}.$$

Obviously, the corresponding function spaces $\mathbf{H}$, $\mathbf{H}^{(n)}$, and $\mathbf{H}^{(m)}$ are related as follows:

$$\mathbf{H}^{(n)} \subset \mathbf{H} \subset \mathbf{H}^{(m)}.$$

Indeed, any element $\mathbf{h}^n \in \mathbf{H}^{(n)}$, extended by zero, lies in $\mathbf{H} \cap \mathbf{H}^{(m)}$, while any element $\mathbf{h} \in \mathbf{H}$ lies in $\mathbf{H}^{(m)}$. Thus we have the relations

$$\Omega_k^{(n)} \geq \Omega_k \geq \Omega_k^{(m)}, \quad \forall\, n, m, \qquad (k = 1, 2, \ldots)$$

where $\Omega_k^{(n)}$ and $\Omega_k^{(m)}$ are eigenfrequencies for the contours ℓ_n and ℓ_m, respectively. Moreover, in this case we also find that

$$\lim_{n\to\infty} \Omega_k^{(n)} = \Omega_k, \quad \lim_{m\to\infty} \Omega_k^{(m)} = \Omega_k \qquad (k = 1, 2, \ldots).$$

Shells with boundary reinforcements

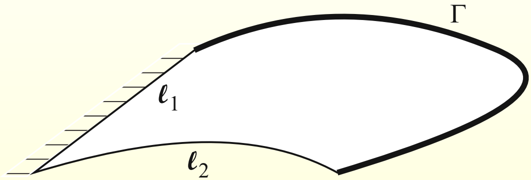


Figure: Shell with boundary reinforcement

Shells with boundary reinforcements

For simplicity, we neglect the inertial properties of the reinforcement but do include tangential, bending, and torsional stiffness parameters. The corresponding boundary conditions are

$$\mathbf{W}\big|_{\ell_1} = \mathbf{0}, \quad \boldsymbol{\Psi}\big|_{\ell_1} = \mathbf{0}, \quad \boldsymbol{\eta}\cdot\boldsymbol{\Phi}_1\big|_{\ell_2} = \mathbf{0}, \quad \boldsymbol{\eta}\cdot\boldsymbol{\Phi}_2\big|_{\ell_2} = \mathbf{0},$$

$$\mathbf{W}\big|_{\Gamma} = \mathbf{W}_\Gamma, \quad \boldsymbol{\Psi}\big|_{\Gamma} = \boldsymbol{\Psi}_\Gamma,$$

and

$$\mathbf{t}' + \boldsymbol{\eta}\cdot\boldsymbol{\Phi}_1\big|_{\Gamma} = \mathbf{0}, \quad \mathbf{m}' + \boldsymbol{\rho}'\big|_{\Gamma}\times\mathbf{t} + \boldsymbol{\eta}\cdot\boldsymbol{\Phi}_2\big|_{\Gamma} = \mathbf{0},$$

where $\mathbf{W}_\Gamma$ and $\boldsymbol{\Psi}_\Gamma$ are the translation and rotation vectors characterizing the Cosserat curve Γ, $\mathbf{t}$ and $\mathbf{m}$ are the stress resultant and stress couple vectors, respectively, acting in the reinforcements, and a prime denotes differentiation with respect to the arc-length parameter s along Γ.

Shells with boundary reinforcements

Finally, we can prove that

$$\Omega_k^\circ \geq \Omega_k^\Gamma \geq \Omega_k^f, \qquad (k = 1, 2, \ldots)$$

where Ω_k°, Ω_k^Γ, and Ω_k^f are the kth eigenfrequencies of shells having the same shape but with contour Γ clamped, elastically reinforced, and free, respectively.

Shells with junctions

Pietraszkiewicz and Konopińska (2011) suggested a classification of junctions for multifolded shells. Figure depicts a shell with three branches.

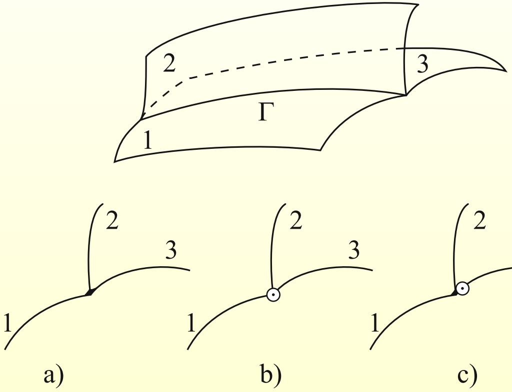


Figure: Multifolded shell and various junctions: (a) stiff junction; (b) entirely simple supported junction; (c) partially simple supported junction

Shells with junctions

For the jth branch ($j = 1, 2, 3$) emanating from the junction, we denote by $\mathbf{W}_j$ and $\boldsymbol{\Psi}_j$ the point translations and rotations, respectively. For any type of junction, the translations are assumed to match along Γ:

$$\mathbf{W}_1\big|_\Gamma = \mathbf{W}_2\big|_\Gamma = \mathbf{W}_3\big|_\Gamma.$$

For the stiff junction of Fig. 3(a), the rotations are also continuous:

$$\boldsymbol{\Psi}_1\big|_\Gamma = \boldsymbol{\Psi}_2\big|_\Gamma = \boldsymbol{\Psi}_3\big|_\Gamma.$$

For the partially simple supported junction of Fig. 3(c), we have

$$\boldsymbol{\Psi}_1\big|_\Gamma = \boldsymbol{\Psi}_2\big|_\Gamma.$$

For the entirely simple supported junction of Fig. 3(b), on the other hand, there are no constraints on $\boldsymbol{\Psi}_j$. For all of these junctions, Courant's principle easily yields

$$\widetilde{\Omega}_k \geq \widehat{\Omega}_k \geq \overline{\Omega}_k \qquad (k = 1, 2, \ldots)$$

Conclusions

- We formulated the Rayleigh variational principle for eigen-oscillations of the prestressed elastic shells.
- Using the Rayleigh quotient we shown that influence of initial stresses on the eigen-frequencies are determined by the changes of the elastic moduli tensors due to deformations and by the term depending on initial stress and couple stress tensors only.
- The latter term may play more important role in the case of flexible thin shells.
- Some specific examples are considered.

References

- Altenbach, H., Eremeyev, V.A.: On the effective stiffness of plates made of hyperelastic materials with initial stresses. *International Journal of Non-Linear Mechanics* **45**(10), 976–981 (2010)
- Altenbach, H., Eremeyev, V.A.: Vibration analysis of non-linear 6-parameter prestressed shells. *Meccanica* **49**, 1751–1761 (2014)

Thank you for your attention!

Further Questions:

eremeyev.victor@gmail.com