The Nonlinear Theory of Thermoelastic Shells Undergoing Phase Transitions

Victor A. Eremeyev

Rzeszow University of Technology, Rzeszów, Poland

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Contents

Introduction

Phase transition (PT) phenomenon in continuous media originally described by Gibbs in 1875–1878 was developed in a number of papers summarised in several books for example by

- Grinfeld (1991)
- Romano (1993)
- Gurtin (1993, 2002)
- Sun (2002)
- Bhattacharya (2003)
- Fischer (2004)
- Abeyaratne & Knowles (2006)
- Lagoudas (2008)
- Berezovski, Engelbrecht & Maugin (2008)

Introduction. PT in shell-like structures

The non-linear equilibrium conditions of elastic shells undergoing phase transition (PT) of martensitic type were formulated by Eremeyev & Pietraszkiewicz (2004, 2009, 2010, 2011), Pietraszkiewicz et al. (2007).

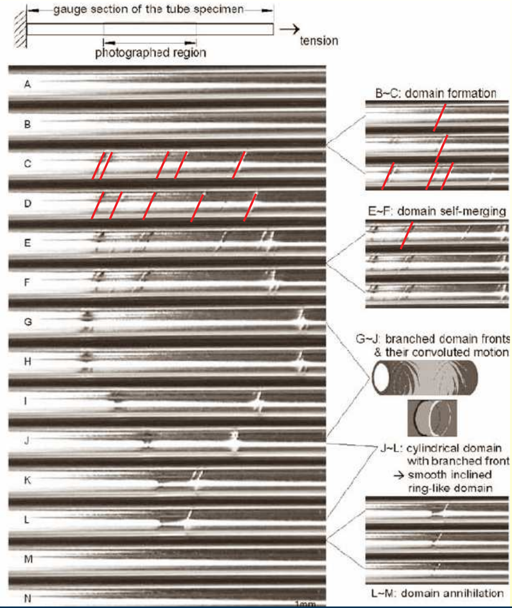
Here we use the general 6-th parametric theory of shells presented in

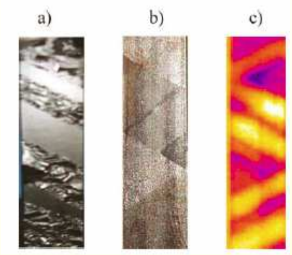
- Libai & Simmonds (1983, 1998)
- Chróścielewski, Makowski & Pietraszkiewicz (2004)
- Eremeyev & Zubov (2008).

Other models of PT in thin-walled structures

- martensitic films (Bhattacharia & James (1999), James & Rizzoni, Shu (2000), and Miyazaki et al. (2009));
- biological membranes (Boulbitch (1999), Agrawal & Steigmann (2008), and Elliott and Stinner (2011));
- other problems (Roytburd & Slutsker (2002)).

PT in tubes and strips





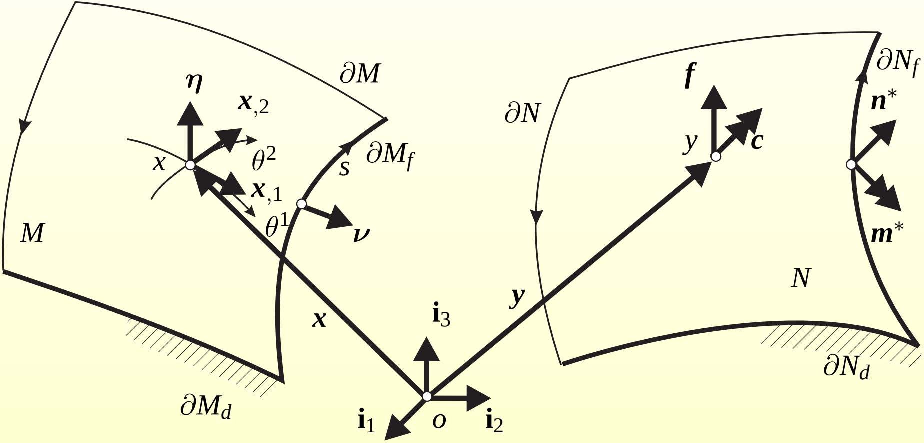
Pieczyska E.A. et al. (2006-2011)

Qing-Ping Sun et al. (2002-2010)

Kinematics

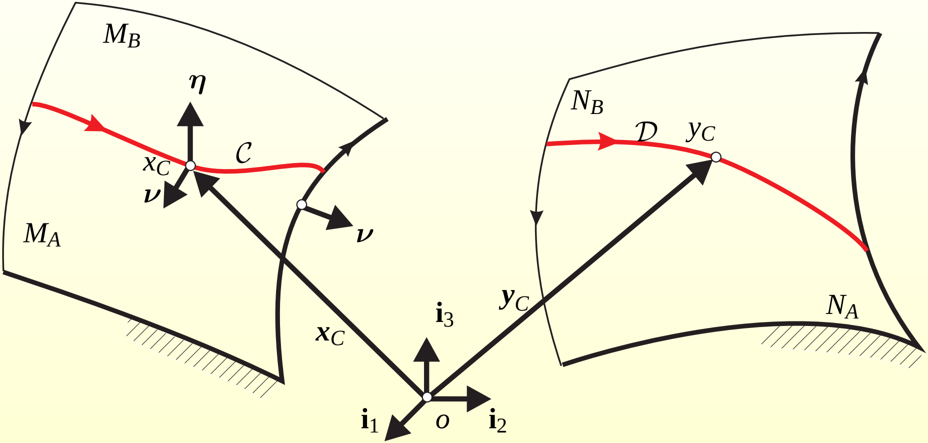
Deformation is given by the following relations

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u}, \quad \boldsymbol{d}_\alpha = \boldsymbol{Q}\boldsymbol{x}_{,\alpha}, \quad \boldsymbol{d} = \boldsymbol{Q}\boldsymbol{n}. \tag{1}$$



The shell with phase interface

Additional variable $\boldsymbol{y}_C$ (or $\boldsymbol{x}_C$)



Equilibrium equations

Global form

$$\mathfrak{F} = \mathbf{0}, \quad \mathfrak{M} = \mathbf{0}, \tag{2}$$

where

$$\begin{aligned}
\mathfrak{F} &= \iint_{\Pi} \boldsymbol{f}\, da + \int_{\partial\Pi\setminus\partial M_f} \boldsymbol{n}_\nu\, ds + \int_{\partial\Pi\cap\partial M_f} \boldsymbol{n}^*\, ds, \\
\mathfrak{M} &= \iint_{\Pi} (\boldsymbol{c} + \boldsymbol{y}\times\boldsymbol{f})\, da + \int_{\partial\Pi\setminus\partial M_f} (\boldsymbol{m}_\nu + \boldsymbol{y}\times\boldsymbol{n}_\nu)\, ds \\
&\quad + \int_{\partial\Pi\cap\partial M_f} (\boldsymbol{m}^* + \boldsymbol{y}\times\boldsymbol{n}^*)\, ds, \\
\boldsymbol{n}_\nu &= \boldsymbol{N}\boldsymbol{\nu}, \qquad \boldsymbol{m}_\nu = \boldsymbol{M}\boldsymbol{\nu}, \qquad \Pi \subset M.
\end{aligned} \tag{3}$$

Local form: the local Lagrangian equilibrium equations and the static boundary conditions are

$$\mathrm{Div}\,\boldsymbol{N} + \boldsymbol{f} = \mathbf{0}, \quad \mathrm{Div}\boldsymbol{M} + \mathrm{ax}\left(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T\right) + \boldsymbol{c} = \mathbf{0} \quad \text{in} \ \ M, \tag{4}$$

$$\boldsymbol{N}\boldsymbol{\nu} - \boldsymbol{n}^* = \mathbf{0}, \quad \boldsymbol{M}\boldsymbol{\nu} - \boldsymbol{m}^* = \mathbf{0} \quad \text{along} \ \ \partial M_f.$$

Compatibility conditions

Static compatibility conditions

$$[\![\boldsymbol{N}\boldsymbol{\nu}]\!] = \mathbf{0}, \quad [\![\boldsymbol{M}\boldsymbol{\nu}]\!] = \mathbf{0} \quad \text{along } \mathcal{C},$$

where $[\![\ldots]\!] = (\ldots)_B - (\ldots)_A$.
There are two types of phase interfaces, the coherent in rotations phase interface, and the incoherent in rotations one, see Eremeyev & Pietraszkiewicz (2004).
For the coherent in rotations phase interface $\mathcal{C}$ we have

$$[\![\boldsymbol{v}]\!] + V[\![\boldsymbol{F}\boldsymbol{\nu}]\!] = \mathbf{0}, \quad [\![\boldsymbol{\omega}]\!] + V[\![\boldsymbol{K}\boldsymbol{\nu}]\!] = \mathbf{0},$$

while for the incoherent in rotations phase interface $\mathcal{C}$ we have

$$[\![\boldsymbol{v}]\!] + V[\![\boldsymbol{F}\boldsymbol{\nu}]\!] = \mathbf{0}.$$

Here $\boldsymbol{v} = \dot{\boldsymbol{u}}$, $\boldsymbol{\omega} = \operatorname{ax}\left(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T\right)$, and $V = \dot{\boldsymbol{x}}_C \cdot \boldsymbol{\nu}$ is the velocity of $\mathcal{C}$.

Thermodynamics of shells. Various 2D formulations

- Direct approach
- Derivation from 3D continuum mechanics

- Murdoch (1976a,b) 1 temperature
- Zhilin (1976) 2 temperatures, axiomatic approach
- Green & Naghdi (1970, 1979) 2 and more temperature fields, axiomatic approach
- Simmonds (1984, 2005, 2011) 2 temperature fields, derivation from 3D
- Makowski & Pietraszkiewicz (2002)
- Pietraszkiewicz (2010)
- Eremeyev & Pietraszkiewicz (2011) 2 temperature fields, derivation from 3D

Energy balance equation

Follow Eremeyev & Pietraszkiewicz (2011) we postulate the energy balance in the form

$$\dot{\mathfrak{E}} = \mathfrak{A} + \mathfrak{Q}, \qquad \mathfrak{E} = \iint\limits_{\Pi} \rho\varepsilon \, da. \tag{5}$$

In (5), $\mathfrak{E}$ the resultant internal energy, $\mathfrak{A}$ the resultant mechanical power, and the resultant heat supply $\mathfrak{Q}$ which are defined as follows

$$\mathfrak{A} = \iint\limits_{\Pi} (\boldsymbol{f}\cdot\boldsymbol{v}+\boldsymbol{c}\cdot\boldsymbol{\omega})\, da + \int\limits_{\partial\Pi\setminus\partial M_f} (\boldsymbol{n}_\nu\cdot\boldsymbol{v}+\boldsymbol{m}_\nu\cdot\boldsymbol{\omega})\, ds + \int\limits_{\partial\Pi\cap\partial M_f} (\boldsymbol{n}^*\cdot\boldsymbol{v}+\boldsymbol{m}^*\cdot\boldsymbol{\omega})\, ds$$

$(\,\boldsymbol{n}_\nu = \boldsymbol{N}\boldsymbol{\nu},\, \boldsymbol{m}_\nu = \boldsymbol{M}\boldsymbol{\nu})$,

$$\mathfrak{Q} = \iint\limits_{\Pi} \rho q \, da - \int\limits_{\partial\Pi\setminus\partial M_h} q_\nu \, ds - \int\limits_{\partial\Pi\bigcap\partial M_h} q^* \, ds.$$

Local form of the energy balance equation

$$\rho\dot{\varepsilon} = \rho q - \mathrm{Div}\,\boldsymbol{q} + \boldsymbol{N} \bullet \boldsymbol{E}^{\circ} + \boldsymbol{M} \bullet \boldsymbol{K}^{\circ} \quad \text{in} \;\; M\backslash\mathcal{C}, \qquad (6)$$

$$\boldsymbol{q} \cdot \boldsymbol{\nu} - q^{*} = 0 \quad \text{along} \;\; \partial M_h,$$

where

$$\boldsymbol{E}^{\circ} = \boldsymbol{Q}\left(\boldsymbol{Q}^{T}\boldsymbol{E}\right)^{\cdot}, \quad \boldsymbol{K}^{\circ} = \boldsymbol{Q}\left(\boldsymbol{Q}^{T}\boldsymbol{K}\right)^{\cdot},$$

$$\boldsymbol{E} = \boldsymbol{\varepsilon}_{\alpha} \otimes \boldsymbol{a}^{\alpha}, \quad \boldsymbol{K} = \boldsymbol{\varkappa}_{\alpha} \otimes \boldsymbol{a}^{\alpha}, \quad \boldsymbol{\varepsilon}_{\alpha} = \boldsymbol{y}_{,\alpha} - \boldsymbol{d}_{\alpha}, \quad \boldsymbol{\varkappa}_{\alpha} = \frac{1}{2}\boldsymbol{d}^{i} \times \boldsymbol{Q}_{,\alpha}\boldsymbol{Q}^{T}\boldsymbol{d}_{i},$$

and $\boldsymbol{A} \bullet \boldsymbol{B} = \mathrm{tr}\,(\boldsymbol{A}^{T}\boldsymbol{B})$.
Introducing the referential shell stress and strain measures

$$\mathbf{N} = \boldsymbol{Q}^{T}\boldsymbol{N}, \quad \mathbf{M} = \boldsymbol{Q}^{T}\boldsymbol{M}, \quad \mathbf{E} = \boldsymbol{Q}^{T}\boldsymbol{E}, \quad \mathbf{K} = \boldsymbol{Q}^{T}\boldsymbol{K},$$

one can rewrite (6) as

$$\rho\dot{\varepsilon} = \rho q - \mathrm{Div}\,\boldsymbol{q} + \mathbf{N} \bullet \dot{\mathbf{E}} + \mathbf{M} \bullet \dot{\mathbf{K}}. \qquad (7)$$

The entropy inequality

Clausius–Duhem inequality has the form

$$\dot{\mathfrak{H}} \geq \mathfrak{J}, \tag{8}$$

where

$$\mathfrak{H} = \iint\limits_{\Pi} \rho\eta \, da, \quad \mathfrak{J} = \iint\limits_{\Pi} \rho j \, da + \int\limits_{\partial\Pi \setminus \partial M_h} j_\nu \, ds + \int\limits_{\partial\Pi \cap \partial M_h} j^* \, ds, \tag{9}$$

$j_\nu = \boldsymbol{j} \cdot \boldsymbol{\nu}$, $\boldsymbol{j} \in T_x\mathcal{M}$.

Shelllike temperature and entropy

Follow Simmonds (2005) we introduce here the shelllike temperature θ and the surface temperature deviation φ

$$\frac{1}{\theta} = \frac{1}{2}\left(\frac{1}{\Theta_+} + \frac{1}{\Theta_-}\right),$$
$$\varphi = \frac{1}{h}\left(\frac{1}{\Theta_-} - \frac{1}{\Theta_+}\right), \tag{10}$$

where $\Theta_\pm > 0$ are temperatures of the upper and lower shell faces $\mathcal{M}^\pm$ taken to be equal to those prevailing in the adjoining external media, and h is the shell thickness.
Two fields θ and φ require also two respective dual fields, which are the resultant surface entropy density η and the surface entropy deviation χ.

The local entropy inequality

Unlike in the 3D entropy balance, the resultant entropy supply j and the resultant entropy flux $\boldsymbol{j}$ take now the extended form Eremeyev & Pietraszkiewicz (2011),

$$\boldsymbol{j} = \frac{1}{\theta} r - \varphi s, \quad \boldsymbol{j} = \frac{1}{\theta}\boldsymbol{q} - \varphi \boldsymbol{s}, \tag{11}$$

where s is the resultant extra heat supply and $\boldsymbol{s}$ is the resultant extra heat flux vector.

Local entropy inequality

$$\rho\dot{\eta} - \rho\left(\frac{r}{\theta} - \varphi s\right) + \frac{1}{\theta}\,\mathrm{Div}\,\boldsymbol{q} - \varphi \mathrm{Div}\,\boldsymbol{s} + \boldsymbol{h}\cdot\boldsymbol{s} - \frac{1}{\theta^2}\,\boldsymbol{q}\cdot\boldsymbol{g} \geq 0 \quad \text{in} \quad \mathcal{M}\backslash\mathcal{C},$$

$$\boldsymbol{g} = \mathrm{Grad}\,\theta, \quad \boldsymbol{h} = \mathrm{Grad}\,\varphi, \qquad \boldsymbol{g}, \boldsymbol{h} \in T_x\mathcal{M}.$$

The local energy and entropy balances at the phase interface $\mathcal{C}$

Local energy balance

$$V[\![\rho\varepsilon]\!] + [\![\boldsymbol{n}_\nu \cdot \boldsymbol{v}]\!] + [\![\boldsymbol{m}_\nu \cdot \boldsymbol{\omega}]\!] - [\![\boldsymbol{q} \cdot \boldsymbol{\nu}]\!] = 0.$$

Local entropy inequalities

$$V[\![\rho\eta]\!] - \left[\!\!\left[\frac{1}{\theta}\boldsymbol{q} \cdot \boldsymbol{\nu}\right]\!\!\right] + [\![\varphi\boldsymbol{s} \cdot \boldsymbol{\nu}]\!] \equiv \delta^2_{\mathcal{C}} \geq 0\,. \qquad (12)$$

Thermoelastic shells

The constitutive equations for thermoelastic shells which take the form Eremeyev & Pietraszkiewicz (2009, 2011),

$$
\begin{gathered}
\psi \equiv \varepsilon - \theta\eta - \varphi\chi = \psi(\boldsymbol{E}, \boldsymbol{K}, \theta, \varphi), \\
\boldsymbol{N} = \rho\psi_{,\boldsymbol{E}}, \quad \boldsymbol{M} = \rho\psi_{,\boldsymbol{K}}, \quad \eta = -\psi_{,\theta}, \quad \chi = -\psi_{,\varphi}, \\
\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{E}, \boldsymbol{K}, \theta, \boldsymbol{g}, \varphi, \boldsymbol{h}), \quad \boldsymbol{s} = \boldsymbol{s}(\boldsymbol{E}, \boldsymbol{K}, \theta, \boldsymbol{g}, \varphi, \boldsymbol{h}),
\end{gathered}
\tag{13}
$$

where we have introduced the surface free energy density ψ.
For thermoelastic shells the local energy balance equation and the local entropy inequality reduce to the form

$$\rho(\theta\dot{\eta} + \varphi\dot{\chi}) = \rho r - \operatorname{Div} \boldsymbol{q}, \tag{14}$$

$$-\rho\dot{\chi} + \rho\theta s - \theta \operatorname{Div} \boldsymbol{s} = c\varphi, \quad c \geq 0, \tag{15}$$

where the new constitutive function c is introduced, and the reduced dissipation inequality becomes

$$-\frac{1}{\theta}\boldsymbol{g} \cdot \boldsymbol{q} - \theta \boldsymbol{h} \cdot \boldsymbol{s} \geq 0. \tag{16}$$

Local balance equations along the phase interface

- Kinematic compatibility conditions

$$[\![\boldsymbol{v}]\!] + V[\![\boldsymbol{F}\boldsymbol{\nu}]\!] = \boldsymbol{0}, \quad [\![\boldsymbol{\omega}]\!] + V[\![\boldsymbol{K}\boldsymbol{\nu}]\!] = \boldsymbol{0}. \tag{17}$$

- Lagrangian dynamic compatibility conditions

$$[\![\boldsymbol{N}\boldsymbol{\nu}]\!] = \boldsymbol{0}, \quad [\![\boldsymbol{M}\boldsymbol{\nu}]\!] + [\![\boldsymbol{y} \times \boldsymbol{N}\boldsymbol{\nu}]\!] = \boldsymbol{0}. \tag{18}$$

- Local energy balance equation

$$V[\![\rho\varepsilon]\!] + [\![\boldsymbol{n}_\nu \cdot \boldsymbol{v}]\!] + [\![\boldsymbol{m}_\nu \cdot \boldsymbol{\omega}]\!] - [\![\boldsymbol{q} \cdot \boldsymbol{\nu}]\!] = 0. \tag{19}$$

- Local entropy inequality

$$V[\![\rho\eta]\!] - \left[\!\!\left[\frac{\boldsymbol{q} \cdot \boldsymbol{\nu}}{\theta}\right]\!\!\right] - [\![\varphi\boldsymbol{p} \cdot \boldsymbol{\nu}]\!] \geq 0. \tag{20}$$

- Local entropy deviation balance equation

$$V\frac{1}{\theta}[\![\rho\chi]\!] - [\![\boldsymbol{s} \cdot \boldsymbol{\nu}]\!] = 0\,. \tag{21}$$

- $[\![\theta]\!] = 0, \quad [\![\varphi]\!] = 0.$

Kinetic equation

From these relations we obtain the compatibility condition in the form

$$\theta \delta^2_{\mathcal{C}} = -V \left\{ [\![\rho\psi]\!] - \boldsymbol{\nu} \cdot \mathbf{N}^T [\![\mathbf{F}\boldsymbol{\nu}]\!] - \boldsymbol{\nu} \cdot \mathbf{M}^T [\![\mathbf{K}\boldsymbol{\nu}]\!] \right\} \quad \text{along } \mathcal{C} \tag{22}$$

for the coherent phase interface, and

$$\theta \delta^2_{\mathcal{C}} = -V \left\{ [\![\rho\psi]\!] - \boldsymbol{\nu} \cdot \mathbf{N}^T [\![\mathbf{F}\boldsymbol{\nu}]\!] \right\} \quad \text{along } \mathcal{C} \tag{23}$$

for the phase interface incoherent in rotations.
The entropy production $\delta^2_{\mathcal{C}}$ remains always non-negative for all thermomechanical processes. This allows us to postulate the kinetic equation, describing motion of the phase interface for all quasistatic processes.

Kinetic equation

We postulate the kinetic equation, describing motion of the phase interface for all quasistatic processes, in the form

$$V = -\mathcal{F}\left(\boldsymbol{\nu} \cdot [\![\boldsymbol{C}]\!]\boldsymbol{\nu}\right), \quad \boldsymbol{C} = \rho\psi\mathbf{A} - \boldsymbol{N}^T\boldsymbol{F} - \boldsymbol{M}^T\boldsymbol{K}, \tag{24}$$

where $\mathcal{F}$ is the non-negative definite kinetic function depending on the jump of $\boldsymbol{C}$ at $\mathcal{C}$, i.e. $\mathcal{F}(\varsigma) \geq 0$ for $\varsigma > 0$, and $\mathbf{A} = \mathbf{1} - \boldsymbol{n} \otimes \boldsymbol{n}$.
After Berezovski et al (2008), we assume $\mathcal{F}(\varsigma)$ in the form

$$\mathcal{F}(\varsigma) = \begin{cases} \dfrac{k(\varsigma - \varsigma_0)}{1 + a(\varsigma - \varsigma_0)} & \varsigma \geq \varsigma_0, \\ 0 & -\varsigma_0 < \varsigma < \varsigma_0, \\ \dfrac{k(\varsigma + \varsigma_0)}{1 - a(\varsigma + \varsigma_0)} & \varsigma \leq -\varsigma_0. \end{cases} \tag{25}$$

Here ς_0 describes the effects associated with nucleation of the new phase and action of the surface tension, a is a parameter describing the limit value of the phase transition velocity, and k is a positive kinetic factor.

One Clausius-Duhem inequality. One temperature

By Murdoch (1976)

$$\frac{d}{dt}\iint\limits_{\Pi}\rho\eta\,da \geq \iint\limits_{\Pi}\rho\left(\frac{q^+}{T^+_{ext}}+\frac{q^-}{T^-_{ext}}+\frac{q_\Pi}{T}\right)da - \int\limits_{\partial\Pi}\frac{q_\nu}{T}\,ds$$

Local form

$$\rho\frac{d\eta}{dt} \geq \rho\left(\frac{q^+}{T^+_{ext}}+\frac{q^-}{T^-_{ext}}+\frac{q_\Pi}{T}\right) - \mathrm{Div}_s\frac{1}{T}\boldsymbol{q}$$

Jump condition at C,

$$V[\![\rho\eta]\!] - \left[\!\!\left[\frac{1}{T}\boldsymbol{q}\cdot\boldsymbol{\nu}\right]\!\!\right] \equiv \delta^2 \geq 0$$

Entropy inequalities. Two temperatures

By Zhilin (1976)

$$\frac{d}{dt}\iint\limits_{\Pi}\rho\eta_{+}da \geq \iint\limits_{\Pi}\rho\left(\frac{q^{+}}{T_{+}^{ext}}+\frac{Q_{+}}{T_{-}}+\frac{q_{\Pi}^{+}}{T_{+}}\right)da - \int\limits_{\partial\Pi\setminus\partial M_h}\frac{q_{\nu}^{+}}{T_{+}}ds,$$

$$\frac{d}{dt}\iint\limits_{\Pi}\rho\eta_{-}\mathrm{d}\,a \geq \iint\limits_{\Pi}\rho\left(\frac{q^{-}}{T_{-}^{ext}}+\frac{Q_{-}}{T_{+}}+\frac{q_{\Pi}^{-}}{T_{-}}\right)da - \int\limits_{\partial\Pi\setminus\partial M_h}\frac{q_{\nu}^{-}}{T_{-}}ds$$

$$\rho\frac{d\eta_{+}}{dt} \geq \rho\left(\frac{q^{+}}{T_{+}^{ext}}+\frac{Q_{+}}{T_{-}}+\frac{q_{\Pi}^{+}}{T_{+}}\right) - \frac{1}{T_{+}}\mathrm{Div}_s\boldsymbol{q}_{+} + \frac{1}{T_{+}^{2}}(\mathrm{Grad}_s T_{+})\cdot\boldsymbol{q}_{+},$$

$$\rho\frac{d\eta_{-}}{dt} \geq \rho\left(\frac{q^{-}}{T_{-}^{ext}}+\frac{Q_{-}}{T_{+}}+\frac{q_{\Pi}^{-}}{T_{-}}\right) - \frac{1}{T_{-}}\mathrm{Div}_s\boldsymbol{q}_{-} + \frac{1}{T_{-}^{2}}(\mathrm{Grad}_s T_{-})\cdot\boldsymbol{q}_{-},$$

$$V[\![\rho\eta_{\pm}]\!] - \left[\!\!\left[\frac{1}{T_{\pm}}\boldsymbol{q}_{\pm}\cdot\boldsymbol{\nu}\right]\!\!\right] \equiv \delta_{\pm}^{2} \geq 0.$$

Infinitesimal deformations

In such a case the strain vectors $\boldsymbol{\varepsilon}_\alpha$ and $\boldsymbol{\varkappa}_\alpha$ are given by Chroscielewski et al. (2004),

$$\boldsymbol{\varepsilon}_\alpha = \boldsymbol{u}_{,\alpha} - \boldsymbol{\vartheta} \times \boldsymbol{x}_{,\alpha}, \quad \boldsymbol{\varkappa}_\alpha = \boldsymbol{\vartheta}_{,\alpha}, \tag{26}$$

where $\boldsymbol{\vartheta}$ is the infinitesimal rotation vector such that

$$\boldsymbol{Q} \approx \mathbf{1} + \boldsymbol{\vartheta} \times \mathbf{1} \quad \text{if} \quad \|\boldsymbol{\vartheta}\| \ll 1.$$

Hence, the strain measures are given by

$$\mathbf{E} = \mathrm{Grad}\, \boldsymbol{u} - \boldsymbol{\vartheta} \times \mathbf{1}, \quad \mathbf{K} = \mathrm{Grad}\, \boldsymbol{\vartheta}. \tag{27}$$

Note that in such a case we approximately have

$$\boldsymbol{N} \cong \mathbf{N}, \quad \boldsymbol{M} \cong \mathbf{M}, \quad \boldsymbol{E} \cong \mathbf{E}, \quad \boldsymbol{K} \cong \mathbf{K}.$$

Constitutive equations

Let us consider the following constitutive equations for the phases A, B:

$$
\begin{aligned}
2\rho\psi^{A,B} = \alpha_1^{A,B}\mathrm{tr}^2\tilde{\mathbf{E}}_{\|} + \alpha_2^{A,B}\mathrm{tr}\,\tilde{\mathbf{E}}_{\|}^2 + \alpha_3^{A,B}\mathrm{tr}\left(\tilde{\mathbf{E}}_{\|}^T\tilde{\mathbf{E}}_{\|}\right) + \alpha_4^{A,B}\boldsymbol{\eta}\cdot\mathbf{E}\mathbf{E}^T\boldsymbol{\eta} \\
+\beta_1^{A,B}\mathrm{tr}^2\mathbf{K}_{\|} + \beta_2^{A,B}\mathrm{tr}\,\mathbf{K}_{\|}^2 + \beta_3^{A,B}\mathrm{tr}\left(\mathbf{K}_{\|}^T\mathbf{K}_{\|}\right) + \beta_4^{A,B}\boldsymbol{\eta}\cdot\mathbf{K}\mathbf{K}^T\boldsymbol{\eta} \\
+\alpha(\theta-\theta_0)\mathrm{tr}\,\mathbf{E} + \beta\varphi\mathrm{tr}\,\mathbf{K} + 2\rho\psi_0^{A,B}(\theta,\varphi).
\end{aligned}
\tag{28}
$$

Here $\tilde{\mathbf{E}} = \mathbf{E} - \mathbf{E}_p^{A,B}$, $\mathbf{E}_p^{A,B} = \epsilon_p^{A,B}\mathbf{A}$, $\epsilon_p^B = 0$, $\mathbf{E}_{\|} = \mathbf{A}\mathbf{E} \in T_xM\otimes T_xM$,
$\mathbf{K}_{\|} = \mathbf{A}\mathbf{K} \in T_xM\otimes T_xM$.
The function (28) generates

$$
\begin{aligned}
&\mathbf{N} = \alpha_1\boldsymbol{A}\,\mathrm{tr}\,\tilde{\mathbf{E}}_{\|} + \alpha_2\tilde{\mathbf{E}}_{\|}^T + \alpha_3\tilde{\mathbf{E}}_{\|} + \alpha_4\boldsymbol{\eta}\otimes\mathbf{E}^T\boldsymbol{\eta} + \alpha(\theta-\theta_0)\mathbf{A}, \\
&\mathbf{M} = \beta_1\boldsymbol{A}\,\mathrm{tr}\,\mathbf{K}_{\|} + \beta_2\mathbf{K}_{\|}^T + \beta_3\mathbf{K}_{\|} + \beta_4\boldsymbol{\eta}\otimes\mathbf{K}^T\boldsymbol{\eta} + \beta\varphi\mathbf{A}, \\
&\rho\eta = -\alpha\mathrm{tr}\,\mathbf{E} - \rho\psi_{0,\theta}, \quad \rho\chi = -\beta\mathrm{tr}\,\mathbf{K} - \rho\psi_{0,\varphi}.
\end{aligned}
\tag{29}
$$

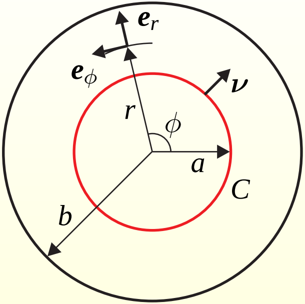
Constitutive parameters

In Chroscielewski et al. (2004) the following relations for the elastic moduli were used:

$$\begin{gathered}
\alpha_1 = C\nu, \quad \alpha_2 = 0, \quad \alpha_3 = C(1-\nu), \quad \alpha_4 = \alpha_s C(1-\nu), \\
\beta_1 = D\nu, \quad \beta_2 = 0, \quad \beta_3 = D(1-\nu), \quad \beta_4 = \alpha_t D(1-\nu), \\
C = \frac{Eh}{1-\nu^2}, \quad D = \frac{Eh^3}{12(1-\nu^2)},
\end{gathered} \tag{30}$$

where E and ν are the Young modulus and the Poisson ratio of the bulk material, respectively, α_s and α_t are dimensionless coefficients, while h is the shell thickness.

Deformation of circular plate undergoing PT



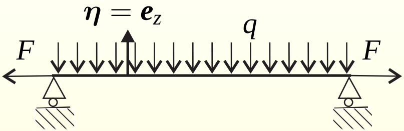


Figure: Two-phase circular plate in the reference configuration

$$\boldsymbol{u} = u(r)\boldsymbol{e}_r + w(r)\boldsymbol{e}_z, \quad \boldsymbol{\vartheta} = \vartheta(r)\boldsymbol{e}_\phi,$$

$$\mathbf{E} = u'\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \frac{u}{r}\boldsymbol{e}_\phi \otimes \boldsymbol{e}_\phi + (w' - \vartheta)\boldsymbol{e}_z \otimes \boldsymbol{e}_r, \quad \mathbf{K} = \vartheta'\boldsymbol{e}_\phi \otimes \boldsymbol{e}_r - \frac{\vartheta}{r}\boldsymbol{e}_r \otimes \boldsymbol{e}_\phi,$$

$$\mathbf{N} = N_{rr}\boldsymbol{e}_r \otimes \boldsymbol{e}_r + N_{\phi\phi}\boldsymbol{e}_\phi \otimes \boldsymbol{e}_\phi + N_{zr}\boldsymbol{e}_z \otimes \boldsymbol{e}_r, \quad \mathbf{M} = M_{\phi r}\boldsymbol{e}_\phi \otimes \boldsymbol{e}_r + M_{r\phi}\boldsymbol{e}_r \otimes \boldsymbol{e}_\phi.$$

Equilibrium equations

The equilibrium equations (4) reduce to three ordinary differential equations

$$N'_{rr} + \frac{1}{r}(N_{rr} - N_{\phi\phi}) + f = 0, \quad N'_{zr} + \frac{1}{r}N_{zr} + q = 0, \\ M'_{\phi r} + \frac{1}{r}(M_{\phi r} + M_{r\phi}) + c_\phi = 0, \tag{31}$$

where $f = \boldsymbol{f} \cdot \boldsymbol{e}_r$, $q = \boldsymbol{f} \cdot \boldsymbol{e}_z$, $c_\phi = \boldsymbol{c} \cdot \boldsymbol{e}_\phi$, and

$$N_{rr} = \alpha_1\left(u' + \frac{u}{r}\right) + \alpha_3 u' - (2\alpha_1 + \alpha_3)\epsilon_p + \alpha(\theta - \theta_0), \\ N_{\phi\phi} = \alpha_1\left(u' + \frac{u}{r}\right) + \alpha_3\frac{u}{r} - (2\alpha_1 + \alpha_3)\epsilon_p + \alpha(\theta - \theta_0), \\ N_{zr} = \alpha_4(w' - \vartheta), \quad M_{r\phi} = \beta_3\vartheta', \quad M_{\phi r} = -\beta_3\frac{\vartheta}{r}. \tag{32}$$

General solution of axi-symmetric problem

In the case of the constant values of the functions f, q, and c_ϕ, the general solution of this system is given by

$$w = w_0 + \frac{c_1 r^2}{2} + c_2 \ln r - \frac{c_\phi r^3}{9\beta_3} - \frac{q r^4}{64\beta_3} - \frac{q r}{2\alpha_4},$$

$$u = d_1 r + \frac{d_2}{r} - \frac{f r^2}{3(\alpha_1 + \alpha_3)}, \qquad (33)$$

$$\vartheta = c_1 r + \frac{c_2}{r} - \frac{c_\phi r^2}{3\beta_3} - \frac{q r^3}{16\beta_3},$$

where c_1, c_2, d_1, d_2, and w_0 are the integration constants.

Boundary conditions

The boundary conditions for the plate are given by the following relations:

$$N_{rr} = F, \quad N_{zr} = 0, \quad M_{\phi r} = 0 \tag{34}$$

at the external boundary of the plate, i.e. at $r = b$,

$$[\![N_{rr}]\!] = [\![N_{zr}]\!] = 0, \quad [\![M_{\phi r}]\!] = 0, \quad [\![u]\!] = [\![w]\!] = 0, \quad [\![\vartheta]\!] = 0 \quad \text{at } r = a \tag{35}$$

at the coherent phase interface, or

$$[\![N_{rr}]\!] = [\![N_{zr}]\!] = 0, \quad M_{\phi r} = 0, \quad [\![u]\!] = [\![w]\!] = 0, \quad \text{at } r = a \tag{36}$$

at the incoherent in rotations phase interface.
For the assumed loading we have $f = 0$ and $c_\phi = 0$.
Kinetic equation is

$$\frac{da}{dt} = -\mathcal{F}\left(\boldsymbol{\nu} \cdot [\![\mathbf{C}]\!]\boldsymbol{\nu}\right) \equiv -\hat{\mathcal{F}}(a; F, q). \tag{37}$$

Stretching

Let us consider the simplest case when $\varphi = 0$, $q = 0$. In this case one has the plane stress state state with

$$w = 0, \quad \vartheta = 0, \quad u = d_1 r + \frac{d_2}{r}, \qquad N_{zr} = 0, \quad M_{r\phi} = M_{\phi r} = 0.$$

There is no any difference between the coherent phase interface and the incoherent in rotations one.

Two cases:

1. Equilibrium: $\boldsymbol{\nu} \cdot [\![\mathbf{C}]\!] \boldsymbol{\nu} = 0$ at C.
2. Quasi-static deformation:

$$\frac{da}{dt} = -\mathcal{F}\left(\boldsymbol{\nu} \cdot [\![\mathbf{C}]\!] \boldsymbol{\nu}\right) \equiv -\hat{\mathcal{F}}\left(a; F\right) \quad \text{at} \quad C.$$

Stretching. Equilibrium

There exist two solutions

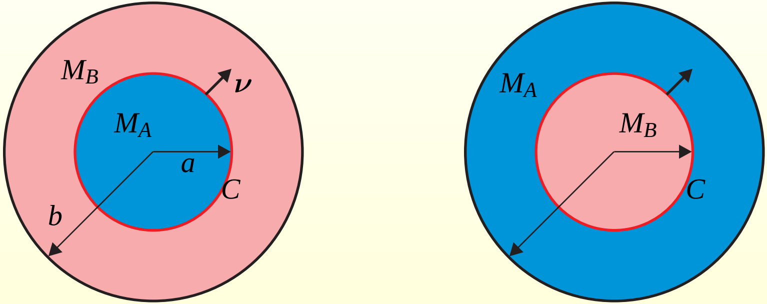


Figure: Two two-phase configurations of a circular plate with one phase boundary

Loading diagram. First solution – one-phase state

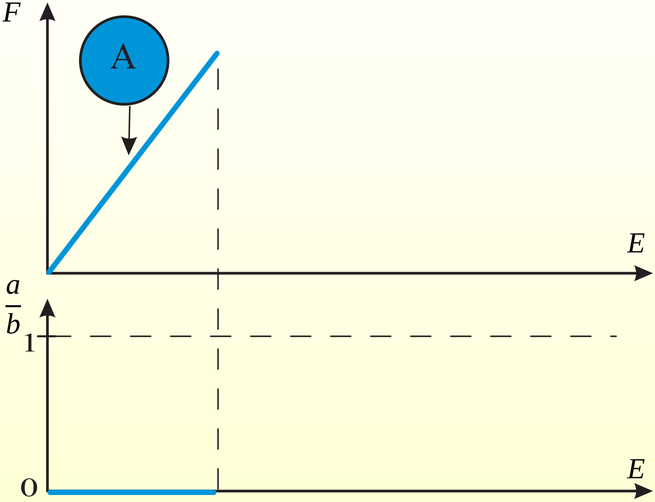


Figure: $F - E$ and $a - E$ diagrams, $E \equiv u(b)/b$

Loading diagram. First solution – two-phase state

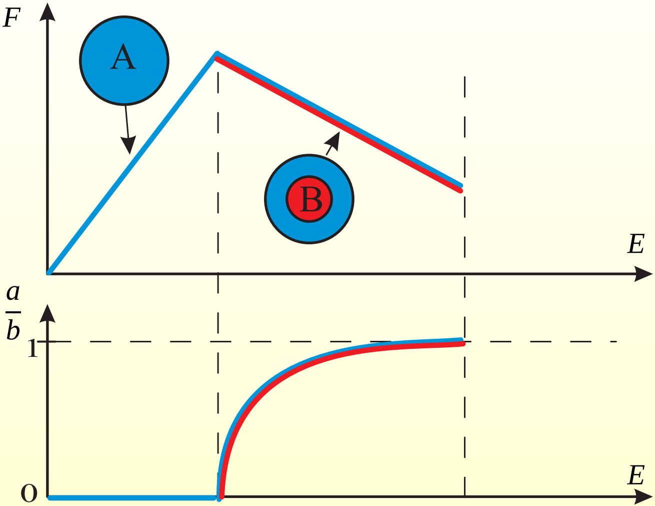


Figure: $F - E$ and $a - E$ diagrams, $E \equiv u(b)/b$

Loading diagram. First solution – one-phase state

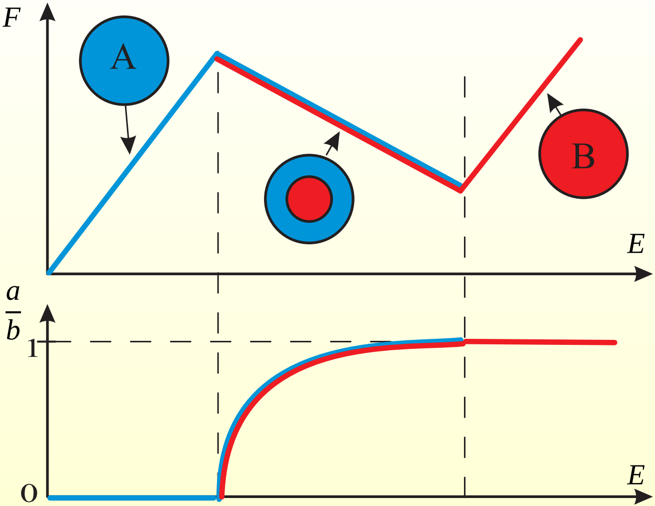


Figure: $F - E$ and $a - E$ diagrams, $E \equiv u(b)/b$

Loading diagram. Second solution

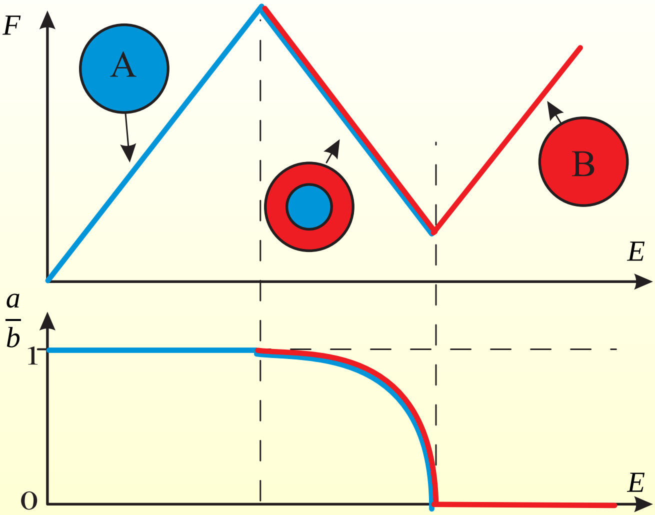


Figure: $F - E$ and $a - E$ diagrams, $E \equiv u(b)/b$

Loading diagram. Both solutions

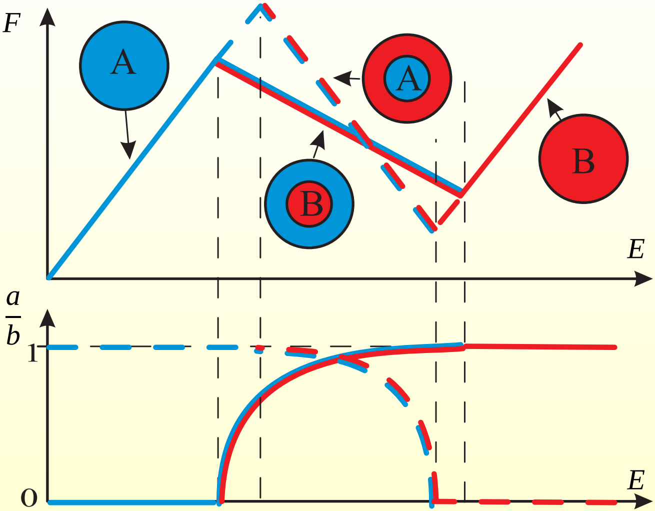


Figure: $F - E$ and $a - E$ diagrams, $E \equiv u(b)/b$

Stretching. Quasistatic deformations (I)

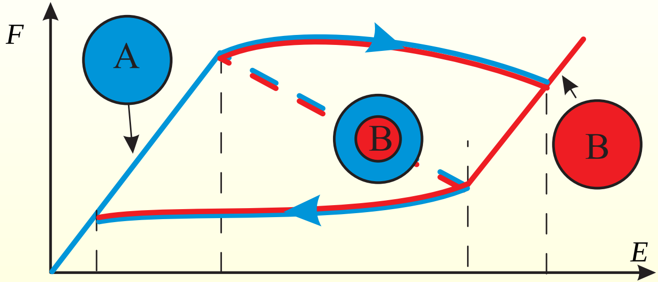


Figure: $F - E$ diagram, $E \equiv u(b)/b$

Simple kinetic function

$$\mathcal{F}(\varsigma) = k\varsigma, \quad k > 0$$

Stretching. Quasistatic deformations (II)

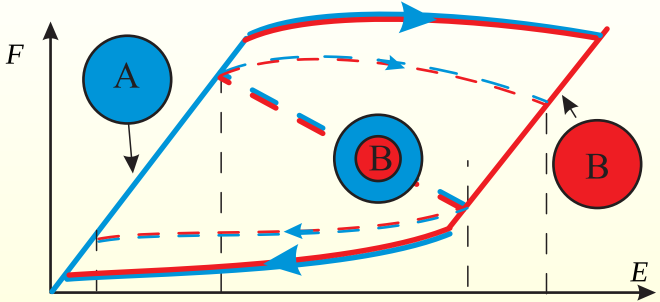


Figure: $F - E$ diagram, $E \equiv u(b)/b$

Kinetic function with nucleation

$$\mathcal{F}(\varsigma) = \begin{cases} k(\varsigma - \varsigma_0) & \varsigma \geq \varsigma_0, \\ 0 & -\varsigma_0 < \varsigma < \varsigma_0, \\ k(\varsigma + \varsigma_0) & \varsigma \leq -\varsigma_0. \end{cases}$$

Stretching. Quasistatic deformations (III)

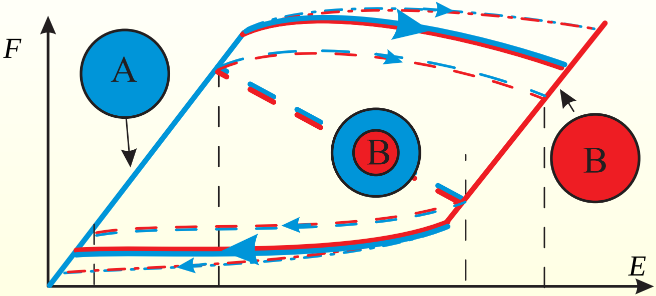


Figure: $F - E$ diagram, $E \equiv u(b)/b$

“General” kinetic function

$$\mathcal{F}(\varsigma) = \begin{cases} k(\varsigma - \varsigma_0)[1 + a(\varsigma - \varsigma_0)]^{-1} & \varsigma \geq \varsigma_0, \\ 0 & -\varsigma_0 < \varsigma < \varsigma_0, \\ k(\varsigma + \varsigma_0)[1 - a(\varsigma + \varsigma_0)]^{-1} & \varsigma \leq -\varsigma_0. \end{cases}$$

Stretching and bending

Displacement and rotation fields

$$w = w_0 + \frac{c_1 r^2}{2} + c_2 \ln r - \frac{qr^4}{64\beta_3} - \frac{qr}{2\alpha_4},$$
$$u = d_1 r + \frac{d_2}{r}, \quad \vartheta = c_1 r + \frac{c_2}{r} - \frac{qr^3}{16\beta_3}.$$

Kinetic equation

$$\frac{da}{dt} = -\hat{\mathcal{F}}(a; F, q). \quad (38)$$

Remark. For the linear theory plates the boundary-value problems for the in-plane deformation and for out-of-plane ones can be solved independently. For the plate with a phase interface the kinetic equation (38) is nonlinear. Hence, for the plates undergoing PT this superposition is impossible, in general.

Stretching and bending. Quasistatic deformations

The action of the transverse load q “deforms” the diagrams of loading which were presented in the case of stretching. The influence of q depends on the values of the elastic moduli α_4 and β_3. For the small enough values of q the $F - E$ diagrams qualitatively coincide with previous ones.

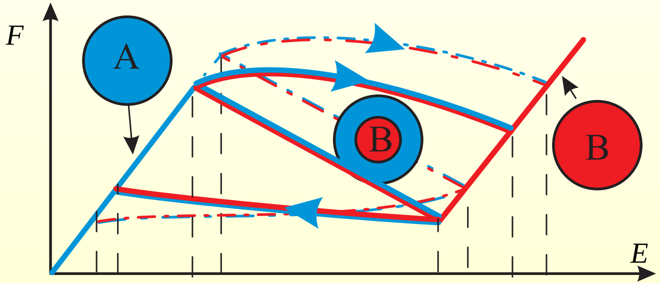
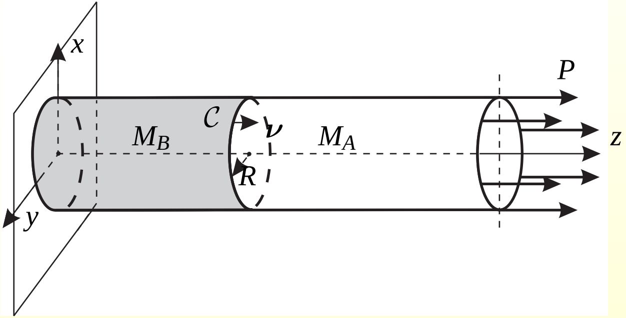


Figure: $F - E$ diagram, $E \equiv u(b)/b$, $q \neq 0$. Simple kinetic function $\mathcal{F}(\varsigma) = k\varsigma, \quad k > 0$

Example: Tension of two-phase tube



Axisymmetric deformation

The axisymmetric deformation state is given by

$$\boldsymbol{u} = u(z)\boldsymbol{e}_z + w(z)\boldsymbol{e}_r, \quad \boldsymbol{\varphi} = \varphi(z)\boldsymbol{e}_\varphi. \tag{39}$$

The discussed example can be reduced to solving the boundary-value problem consisting of the following system of ODEs:

$$\begin{aligned}
&N'_{zz} = 0, \quad N'_{rz} = \tfrac{N_{\phi\phi}}{R}, \quad M'_{\phi z} = -\tfrac{M_{r\phi}}{R} - N_{rz}, \\
&N_{zz} = C(u' - \epsilon_p) + C\nu(w/R - \epsilon_p), \\
&N_{rz} = \alpha_s C(1-\nu)(w' - \varphi), \\
&N_{\phi\phi} = C\nu(u' - \epsilon_p) + C(w/R - \epsilon_p), \\
&M_{\phi z} = D(1-\nu)\varphi', \quad M_{r\phi} = -\alpha_t D(1-\nu)\tfrac{\varphi}{R}, \\
&u(0) = w(0) = \varphi(0) = 0, \\
&N_{zz}(L) = P, \quad N_{rz}(L) = M_{\phi z}(L) = 0,
\end{aligned} \tag{40}$$

where C, D, ν, α_s, and α_t are elastic moduli, and ϵ_p is the phase transformation strain.

Membrane solution

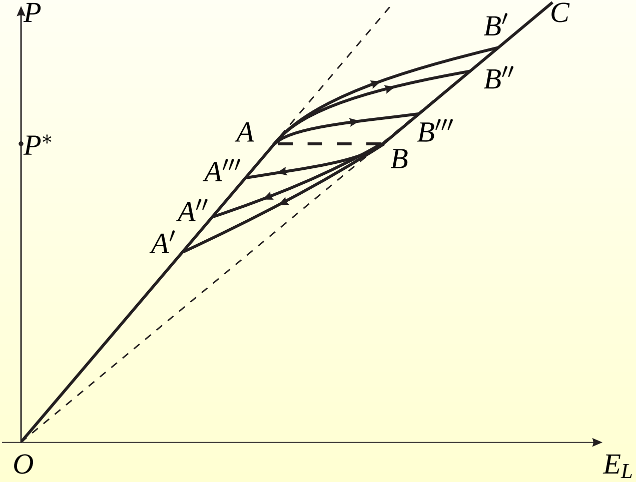
The system (40) has always the particular solution

$$
\begin{aligned}
u(z) &= u_p(z) \equiv \left(\frac{P}{C(1-\nu^2)} + \epsilon_p \right) z + \text{const}, \\
w(z) &= w_p \equiv - \left(\frac{P\nu}{C(1-\nu^2)} - \epsilon_p \right) R, \quad \varphi = 0,
\end{aligned}
\tag{41}
$$

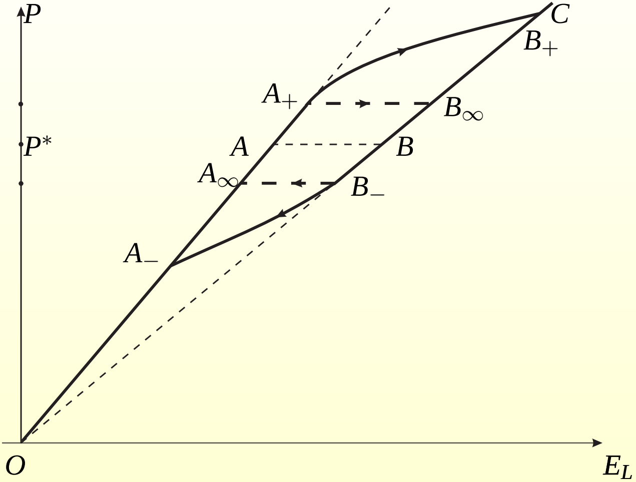
for which $N_{zz} = P$, $N_{rz} = 0$, $N_{\phi\phi} = 0$, $\boldsymbol{M} = \mathbf{0}$. This solution describes the axisymmetric membrane equilibrium state of the cylinder. In the two-phase cylinder such a solution is possible only when $\nu_A = \nu_B = 0$ or $\frac{\nu_A}{C_A(1-\nu_A^2)} = \frac{\nu_B}{C_B(1-\nu_B^2)}$ and $\epsilon_p^A = \epsilon_p^B$.

We first solve the simplest case when $\nu_A = \nu_B = 0$. This problem becomes entirely analogous to the 1D problem discussed by Abeyaratne& Knowles (2006) as a model problem of the 3D continuum model of PT.

$P - E_L$ curves for two-phase shell for different values of $\hat{k}$



$P - E_L$ curves for $\varsigma_0 \neq 0$



General solution

In the general case the solutions of (40) for translation and rotations is more complicated. In particular, for w we obtain

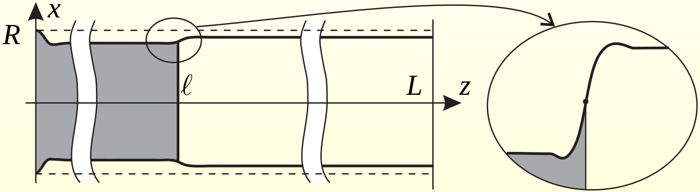
$$\begin{aligned} w &= w_\circ(z) + w_p, \\ w_\circ &= e^{-\omega_R \bar{z}}(c_1 \cos \omega_I \bar{z} + c_2 \sin \omega_I \bar{z}) \\ &\quad + e^{\omega_R \bar{z}}(c_3 \cos \omega_I \bar{z} + c_4 \sin \omega_I \bar{z}), \qquad \bar{z} = z/R, \end{aligned} \tag{42}$$

where c_k, $k = 1, ..., 4$, are integration constants,

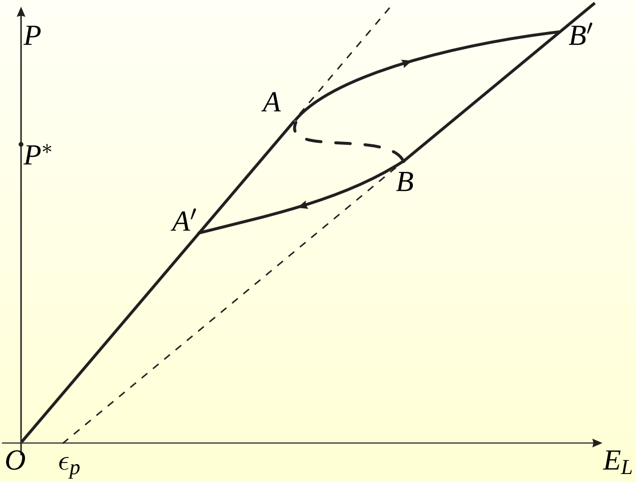
$$\begin{aligned} \{\omega_R, \omega_I\} &= \{\mathrm{Re}, \mathrm{Im}\}\sqrt{2\,\eta_1 + 2i\sqrt{4\,\eta_2 - \eta_1{}^2}}/2, \\ \eta_1 &= \tfrac{\alpha_s \alpha_t + 1 + \nu}{\alpha_s}, \quad \eta_2 = (1+\nu)\left(12\delta^{-2} + \tfrac{\alpha_t}{\alpha_s}\right), \\ \zeta &= (1+\nu)\left(12\delta^{-3} + \alpha_t \delta^{-1}\right), \quad \delta = h/R. \end{aligned}$$

For a thin tube $\eta_2 \gg \eta_1$. Indeed, if one takes $\nu = 1/3$, $\alpha_s = 5/6$, $\alpha_t = 7/10$ and $\delta = 0.1$, then $\eta_1 = 2.3$ and $\eta_2 = 1601.12$, $\omega_R = 4.537$ and $\omega_I = 4.408$, respectively. Hence, we can apply some asymptotic formulae for the boundary layers.

Shape of the thin-walled two-phase tube after phase transition (magnified)



$P - E_L$ curves following from the general solution



Conclusion

- General nonlinear theory thermoelastic shells with PT is discussed
- Local thermodynamic balance equations at the curvilinear interface are derived
- Various formulations of shell thermodynamics are discussed
- Kinetic equation describing quasistatic deformation process of the interface is given
- The mechanics of two-phase plates undergoing phase transitions is presented
- The stretching and bending of a circular two-phase plate is investigated

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Thank you for your attention!!!

Further questions:

eremeyev.victor@gmail.com