

On characterization of an elastic network of orthogonal inextensible fibers with stiff junctions using the six-parameter shell theory

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Outline

- 1 Introduction
- 2 Equilibrium of a beam
- 3 Equilibrium of an elastic network
- 4 Basic equations of nonlinear shells
- 5 On pantographic beam lattice
- 6 Conclusions

Shell and Plate Theories and Its Applications

- Shell-like and plate-like structures are used in **civil and aerospace engineering** as basic elements of constructions. Such structures are applied as a model of analysis in other branches, e.g. **mechanical engineering**, but also in new branches like **medicine**.
- New applications are primarily related to **advanced materials** – instead of steel or concrete, one has to analyze **sandwiches, laminates, foams, nano-films, biological membranes**, etc.
- The new trends in applications demand improvements of the theoretical foundations of the shell theory, since **new effects** must be taken into account. For example, in the case of small-size shell-like structures (nano-tubes) the **surface effect** plays an important role in the mechanical analysis of these structural elements.

Ways of refinement of plate and shell theories

Refinement of 2D governing relations

New more reliable 2D multi-field models may be needed for shells with complex microstructure, to describe non-local and/or complex inelastic behaviour in shells such as fracture and damage, for shells with surface and/or residual stresses, for high-frequency vibrations of shells and wave propagation, etc.

Application to modelling new materials and phenomena

- Thin-walled nanostructures
- Irregular and singular shell problems
- Thin-walled structures in biological, biomechanical and medical sciences
- Nonclassical thin walled composites materials (metamaterials)

Elastic network

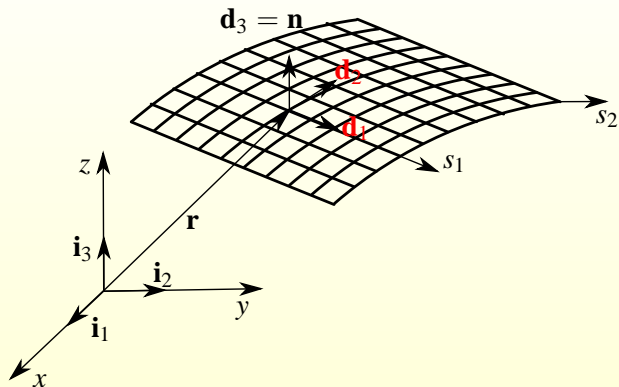


Figure: Elastic network in actual configuration.

Equivalent shell

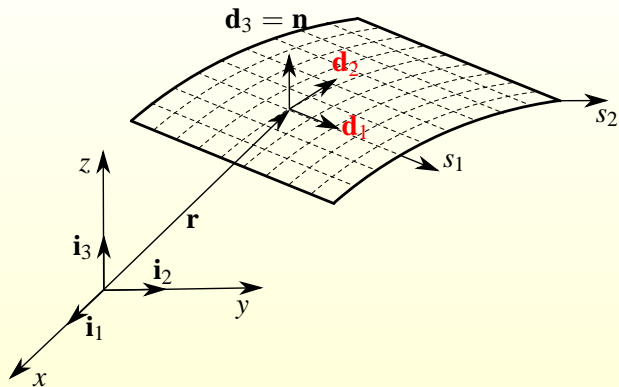


Figure: Shell equivalent to an elastic network.

Cosserat curve kinematics

The position of the directed curve is described by the position vector $\mathbf{R}(s)$ and the attached vectors (directors) $\mathbf{D}_i(s)$, $i = 1, 2, 3$. The unit vectors $\mathbf{D}_i(s)$ are mutually orthogonal and they are chosen such that \mathbf{D}_3 coincides with the unit tangent $\mathbf{t} \equiv \mathbf{R}'$, and $\mathbf{D}_1, \mathbf{D}_2$ belong to the normal plane to the curve \mathcal{C}_0 . The rotations of the attached triad of directors describe the rotations of the rod's cross-sections during deformation.

Let \mathcal{C} be the deformed configuration of the beam at time t , which is characterized by the vector fields

$$\mathbf{r} = \mathbf{r}(s, t), \quad \mathbf{d}_i = \mathbf{d}_i(s, t), \quad i = 1, 2, 3, \quad (1)$$

where \mathbf{r} is the position vector and \mathbf{d}_i are the directors after deformation. Let $\mathbf{P}(s, t) = \mathbf{d}_k(s, t) \otimes \mathbf{D}_k(s)$ be the rotation tensor.

Kinematics

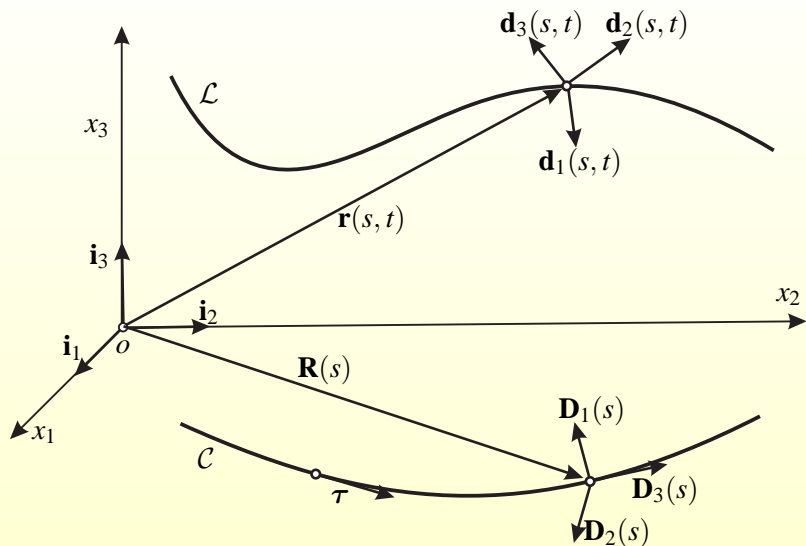


Figure: The actual and reference configurations of a directed curve.

Equilibrium equations

Lagrangian equilibrium equations are

$$\begin{aligned}\mathbf{N}'(s) + \rho_0 \mathbf{f} &= \mathbf{0}, \\ \mathbf{M}'(s) + \mathbf{r}' \times \mathbf{N}(s) + \rho_0 \mathbf{c} &= \mathbf{0},\end{aligned}\tag{2}$$

where \mathbf{N} is the force vector, \mathbf{M} is the moment vector, \mathbf{f} and \mathbf{c} are the external body force and moment per unit mass, ρ_0 is the mass density per unit length of \mathcal{C}_0 .

Constitutive equations

For elastic beams, there exists the strain energy density \mathcal{U} which is a function of the following arguments

$$\mathcal{U} = \mathcal{U}(\mathbf{r}', \mathbf{P}').$$

Applying to this equation the material frame-indifference principle we conclude that \mathcal{U} is invariant under following transformations

$$\mathcal{U}(\mathbf{r}', \mathbf{P}') = \mathcal{U}(\mathbf{Q}\mathbf{r}', \mathbf{Q}\mathbf{P}')$$

for any orthogonal tensor \mathbf{Q} . Using $\mathbf{Q} = \mathbf{P}^T$ we get

$$\mathcal{U} = \mathcal{U}(\mathbf{P}^T \mathbf{r}', \mathbf{P}^T \mathbf{P}').$$

Taking into account that $\mathbf{P}^T \mathbf{P}'$ is skew-symmetric tensor we obtain that

$$\mathbf{P}^T \mathbf{P}' = \mathbf{k} \times \mathbf{I}, \quad \mathbf{k} = \text{axial}(\mathbf{P}^T \mathbf{P}'),$$

where \mathbf{I} is the unit tensor and \mathbf{k} is the vector of bending-twisting of the beam.

Constitutive equations

Replacing $\mathbf{P}^T \mathbf{r}'$ by vector $\mathbf{e} = \mathbf{P}^T \mathbf{r}' - \mathbf{t}$ which is zero on the rigid motions of the beam we come to the constitutive equation in the form

$$\mathcal{U} = \mathcal{U}(\mathbf{e}, \mathbf{k}). \quad (3)$$

In what follows we consider \mathcal{U} as quadratic form of two vectors \mathbf{e} and \mathbf{k} :

$$2\mathcal{U} = \mathbf{e} \cdot \mathbf{K}_s \mathbf{e} + \mathbf{k} \cdot \mathbf{K}_b \mathbf{k}, \quad (4)$$

where \mathbf{K}_1 and \mathbf{K}_2 are stiffness tensors. The stress measures are related with \mathcal{U} by the formulae

$$\mathbf{N} = \frac{\partial \mathcal{U}}{\partial \mathbf{e}} \mathbf{P}^T, \quad \mathbf{M} = \frac{\partial \mathcal{U}}{\partial \mathbf{k}} \mathbf{P}^T.$$

Plane deformations

In order to consider simplest but still nonlinear case and neglecting torsion we restrict ourselves by plane deformations. Here

$$\mathbf{r}' = \lambda(\cos \phi \mathbf{i}_1 + \sin \phi \mathbf{i}_2), \quad \mathbf{P} = \cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{i}_3 \otimes \mathbf{i}_3 + \mathbf{i}_3 \times \mathbf{I} \sin \theta,$$

where λ is stretching, \mathbf{i}_k are Cartesian base vectors, ϕ is the angle between actual tangent to the beam and the horizontal line, and θ is the angle of rotation of the beam cross-section. In addition we consider inextensible beam, so $\lambda = 1$. Then we have

$$\begin{aligned} \mathbf{P}^T \mathbf{r}' &= \cos(\phi - \theta) \mathbf{i}_1 + \sin(\phi - \theta) \mathbf{i}_2, \\ \text{axial}(\mathbf{P}^T \mathbf{P}') &= \mathbf{i}_3 \theta'. \end{aligned}$$

Thus,

$$\mathbf{e} = [\cos(\phi - \theta) - 1] \mathbf{i}_1 + \sin(\phi - \theta) \mathbf{i}_2, \quad \mathbf{k} = \theta' \mathbf{i}_3. \quad (5)$$

Assuming \mathcal{U} as quadratic form (4) we obtain that

$$\underline{2\mathcal{U} = K_1 [\cos(\phi - \theta) - 1]^2 + K_2 \sin^2(\phi - \theta) + K_b \theta'^2}. \quad (6)$$

Plane deformations

If we instead of ϕ introduce the Cartesian components of $\mathbf{r} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2$ then

$$\mathbf{r}' = x'_1\mathbf{i}_1 + x'_2\mathbf{i}_2, \quad \lambda = \sqrt{x_1'^2 + x_2'^2}, \quad \cos \phi = \frac{x'_1}{\sqrt{x_1'^2 + x_2'^2}}, \quad \sin \phi = \frac{x'_2}{\sqrt{x_1'^2 + x_2'^2}}$$

we obtain more complex form of the strain energy.

Remark 1. For the classic beam theory $\phi = \theta$ and for an inextensible beam we have $2\mathcal{U} = K_b\theta'^2$.

Remark 2. Linear theory. For the linear beam theory $|\phi - \theta| \ll 1$ and for small deformations we have

$$2\mathcal{U} = K_2(\phi - \theta)^2 + K_b\theta'^2.$$

Here $\phi = x'_2$.

Constraints and assumptions

- The beam is straight in initial configuration, it is not naturally twisted, etc.
- Inextensibility: $\mathbf{e} \cdot \mathbf{t} = 0$ or $\lambda = 1$.
- It is not shear deformable, $\phi = \theta$ or $\mathbf{e} = \mathbf{0}$.
- \mathbf{d}_3 is tangent to the curve.

Results: Euler–Bernoulli beam

- \mathbf{N} should be determined from the equilibrium equations
- The strain energy has the form $\mathcal{U} = \mathcal{U}(\mathbf{k})$.
- For plane deformation without torsion $\mathcal{U} = \mathcal{U}(\theta')$.

Elastic network

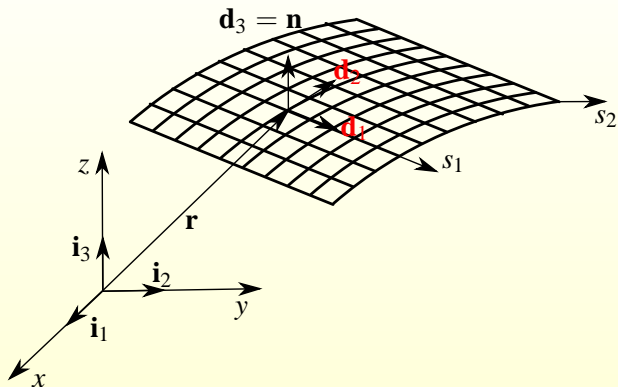


Figure: Elastic network made of identical fibers in actual configuration.

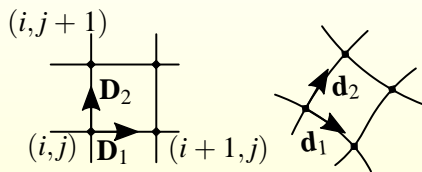
Equilibrium equations

Lagrangian equilibrium equations are

$$\begin{aligned} \mathbf{N}'_{1,1}(s_1) + \rho_0 \mathbf{f}_1 &= \mathbf{0}, \\ \mathbf{M}'_{1,1}(s_1) + \mathbf{d}_1 \times \mathbf{N}_1(s_1) + \rho_0 \mathbf{c}_1 &= \mathbf{0}, \\ \mathbf{N}'_{2,2}(s_2) + \rho_0 \mathbf{f}_2 &= \mathbf{0}, \\ \mathbf{M}'_{2,2}(s_2) + \mathbf{d}_2 \times \mathbf{N}_2(s_2) + \rho_0 \mathbf{c}_1 &= \mathbf{0}, \end{aligned} \tag{7}$$

where \mathbf{N}_α are the force vectors, \mathbf{M}_α are the moment vectors, \mathbf{f}_α and \mathbf{c}_α are the external body forces and moments per unit mass, ρ_0 is the mass density per unit length, and $(\dots)'_{,\alpha} = \frac{\partial(\dots)}{\partial s_\alpha}$.

Deformation of a cell



Total energy

$$\mathcal{E} = \sum_{j=0}^n \sum_{i=0}^{m-1} \int_{s_{1i}}^{s_{1i+1}} \mathcal{U}(s_1) ds_1 + \sum_{i=0}^m \sum_{j=0}^{n-1} \int_{s_{2j}}^{s_{2j+1}} \mathcal{U}(s_2) ds_2, \quad (8)$$

where m and n is the numbers of vertical and horizontal fibers.
Discrete forms are

$$\int_{s_{1i}}^{s_{1i+1}} \mathcal{U}(s_1) ds_1 = \frac{h}{2} [\mathcal{U}(s_{1i}) + \mathcal{U}(s_{1i+1})], \quad (9)$$

$$\int_{s_{2j}}^{s_{2j+1}} \mathcal{U}(s_2) ds_2 = \frac{h}{2} [\mathcal{U}(s_{2j}) + \mathcal{U}(s_{2j+1})]. \quad (10)$$

Thus,

$$\mathcal{E} = \sum_i \sum_j [\mathcal{U}(s_{1i}) + \mathcal{U}(s_{2j})] = \sum_i \sum_j [\mathcal{U}(\mathbf{k}_{1ij}) + \mathcal{U}(\mathbf{k}_{2ij})]. \quad (11)$$

In-plane deformations

The strain energy densities are

$$\mathcal{U}(\mathbf{k}_{1ij}) = \mathcal{U}(\theta'_{1ij,1}), \quad \mathcal{U}(\mathbf{k}_{2ij}) = \mathcal{U}(\theta'_{2ij,2}).$$

Instead of $\theta_1(s_1)$ and $\theta_2(s_2)$ and taking into account that at nodes $\theta_2 = \theta_1 + \pi/2$ we introduce $\theta = \theta(s_1, s_2)$. So,

$$\mathcal{E} = \sum_i \sum_j [\mathcal{U}(\theta'_{ij,1}) + \mathcal{U}(\theta'_{ij,2})]. \quad (12)$$

Continuous analog of (12) is

$$\mathcal{E} = \iint_{\omega} [\mathcal{U}(\theta'_{,1}) + \mathcal{U}(\theta'_{,2})] d\omega. \quad (13)$$

Six-parameter theory of shells

Here we use the [general theory of shells](#) called also [six-parameter theory of shells](#) or [resultant theory of shells](#) or [micropolar theory of shells](#). The characteristic peculiarities are

- The kinematics of the shell is determined by two kinematically independent fields of translations and rotations. 6 degrees of freedom: each point of the micropolar shell base surface has six degrees of freedom as in the rigid body dynamics or in mechanics of micropolar continua.
- At the shell boundary acts forces and moments only.
- The drilling moment is also taken in account.
- Using certain constraints one can reduce the micropolar shell theory to the Kirchhoff–Love or Reissner–Mindlin shell models.

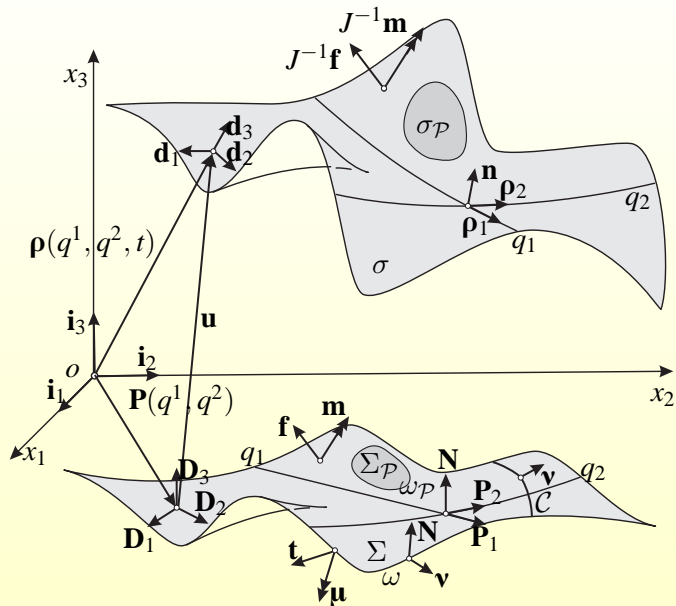
Shell Kinematics

- The deformation of the shell is described by mapping from one state called the *reference configuration* to another one called the *actual configuration*.
- In the reference configuration $\varkappa: \Sigma$ is a base surface with position vector $\mathbf{P}(q^1, q^2)$ and with directors $\mathbf{D}_k(q^1, q^2, \cdot)$, $k = 1, 2, 3$.
- In the actual configuration $\chi: \sigma$ is the base surface with position vector $\boldsymbol{\rho}(q^1, q^2, t)$ and directors $\mathbf{d}_k(q^1, q^2, t)$, $k = 1, 2, 3$.

Hence, the shell is described by two kinematically independent fields

$$\boldsymbol{\rho} = \boldsymbol{\rho}(q^1, q^2, t) \quad \text{and} \quad \mathbf{Q} \equiv \mathbf{d}_k \otimes \mathbf{D}_k = \mathbf{Q}(q^1, q^2, t). \quad (14)$$

Shell Kinematics



Constitutive Equations of Elastic Shells (1)

According to the local action principle ¹, the strain energy density W takes the form

$$W = W(\boldsymbol{\rho}, \nabla_{\mathcal{X}}\boldsymbol{\rho}, \mathbf{Q}, \nabla_{\mathcal{X}}\mathbf{Q}),$$

where

$$\nabla_{\mathcal{X}} \triangleq \mathbf{P}^{\alpha} \frac{\partial}{\partial q^{\alpha}} \quad (\alpha, \beta = 1, 2),$$

$$\mathbf{P}^{\alpha} \cdot \mathbf{P}_{\beta} = \delta_{\beta}^{\alpha}, \quad \mathbf{P}^{\alpha} \cdot \mathbf{N} = 0, \quad \mathbf{P}_{\beta} = \frac{\partial \mathbf{P}}{\partial q^{\beta}}.$$

Here vectors \mathbf{P}_{β} and \mathbf{P}^{α} denote the natural and reciprocal bases on Σ , respectively, \mathbf{N} is the unit normal to Σ , δ_{β}^{α} is the Kronecker symbol, and $\nabla_{\mathcal{X}}$ is the surface nabla operator on Σ .

¹C. Truesdell, W. Noll, The nonlinear field theories of mechanics, in: S. Flügge (Ed.), Handbuch der Physik, Vol. III/3, Springer, Berlin, 1965, pp. 1–602.

Constitutive Equations of Elastic Shells (2)

From the principle of material frame-indifference² it follows that W depends on two surface strain measures \mathbf{E} and \mathbf{K} :

$$W = W(\mathbf{E}, \mathbf{K}), \quad (15)$$

where

$$\mathbf{E} = \mathbf{F} \cdot \mathbf{Q}^T - \mathbf{A}, \quad \mathbf{K} = \frac{1}{2} \mathbf{P}^\alpha \otimes \left(\frac{\partial \mathbf{Q}}{\partial q^\alpha} \cdot \mathbf{Q}^T \right)_\times. \quad (16)$$

Here $\mathbf{F} = \nabla_{\mathcal{X}} \boldsymbol{\rho}$, and $\mathbf{A} \triangleq \mathbf{I} - \mathbf{N} \otimes \mathbf{N}$, \mathbf{I} is the unit 3D tensor,

$$\mathbf{T}_\times = (T^{mn} \mathbf{i}_m \otimes \mathbf{i}_n)_\times = T^{mn} \mathbf{i}_m \times \mathbf{i}_n$$

for any base \mathbf{i}_m , \times denotes the vector (cross) product.

²C. Truesdell, W. Noll, The nonlinear field theories of mechanics, in: S. Flügge (Ed.), Handbuch der Physik, Vol. III/3, Springer, Berlin, 1965, pp. 1–602.

Vectorial Parameterizations of Strain Measures

Introducing

- the translation vector $\mathbf{u} = \boldsymbol{\rho} - \mathbf{P}$ and
- the finite rotation vector $\boldsymbol{\theta} = 2\mathbf{e} \tan \varphi/2$

we can express \mathbf{Q} , \mathbf{E} and \mathbf{K} as follows (see ³ for details)

$$\mathbf{Q} = \frac{1}{(4 + \theta^2)} [(4 - \theta^2)\mathbf{I} + 2\boldsymbol{\theta} \otimes \boldsymbol{\theta} - 4\mathbf{I} \times \boldsymbol{\theta}], \quad (17)$$

$$\mathbf{E} = (\mathbf{A} + \nabla_{\mathcal{X}}\mathbf{u}) \cdot \mathbf{Q}^T - \mathbf{A}, \quad \theta^2 = \boldsymbol{\theta} \cdot \boldsymbol{\theta}, \quad (18)$$

$$\mathbf{K} = \frac{4}{4 + \theta^2} \nabla_{\mathcal{X}}\boldsymbol{\theta} \cdot \left(\mathbf{I} + \frac{1}{2}\mathbf{I} \times \boldsymbol{\theta} \right). \quad (19)$$

\mathbf{Q} describes the rotation about axis with the unit vector \mathbf{e} through an angle φ .

³W. Pietraszkiewicz, V. A. Eremeyev, On vectorially parameterized natural strain measures of the non-linear Cosserat continuum, International Journal of Solids and Structures 46 (11–12) (2009) 2477–2480.

Virtual work principle

Lagrangian equilibrium equations for a micropolar shell can be derived from *the virtual work principle*

$$\delta \iint_{\Sigma} W \, d\Sigma = \delta' A, \quad (20)$$

where

$$\delta' A = \iint_{\Sigma} (\mathbf{f} \cdot \delta \boldsymbol{\rho} + \mathbf{m} \cdot \delta' \boldsymbol{\psi}) \, d\Sigma + \int_{\omega_2} \mathbf{t} \cdot \delta \boldsymbol{\rho} \, ds + \int_{\omega_4} \boldsymbol{\mu} \cdot \delta' \boldsymbol{\psi} \, ds,$$

$$\mathbf{I} \times \delta' \boldsymbol{\psi} = -\mathbf{Q}^T \cdot \delta \mathbf{Q}.$$

In Eq. (20), δ is the symbol of variation, $\delta' \boldsymbol{\psi}$ the virtual rotation vector, \mathbf{f} the surface force density distributed on Σ , and \mathbf{m} the surface couple density distributed on Σ . The quantities \mathbf{t} and $\boldsymbol{\mu}$ are the linear densities of forces and couples distributed along corresponding parts of the shell boundary ω , respectively.

Lagrangian Shell Equations

Using the formulae

$$\begin{aligned}\delta W &= \frac{\partial W}{\partial \mathbf{E}} \bullet \delta \mathbf{E} + \frac{\partial W}{\partial \mathbf{K}} \bullet \delta \mathbf{K}, \\ \delta \mathbf{E} &= (\nabla_s \delta \rho) \cdot \mathbf{Q}^T + \mathbf{F} \cdot \delta \mathbf{Q}^T, \quad \delta \mathbf{K} = (\nabla_s \delta' \psi) \cdot \mathbf{Q}^T, \\ \delta' \psi &= \frac{4}{4 + \theta^2} \left(\delta \boldsymbol{\theta} + \frac{1}{2} \boldsymbol{\theta} \times \delta \boldsymbol{\theta} \right),\end{aligned}$$

and Eq. (20), we obtain the *Lagrangian shell equations*:

$$\nabla_{\mathcal{X}} \cdot \mathbf{T}_{\mathcal{X}} + \mathbf{f} = \mathbf{0}, \quad \nabla_{\mathcal{X}} \cdot \mathbf{M}_{\mathcal{X}} + [\mathbf{F}^T \cdot \mathbf{T}_{\mathcal{X}}]_{\times} + \mathbf{m} = \mathbf{0}, \quad (21)$$

$$\mathbf{T}_{\mathcal{X}} = \mathbf{S}_1 \cdot \mathbf{Q}, \quad \mathbf{M}_{\mathcal{X}} = \mathbf{S}_2 \cdot \mathbf{Q}, \quad \mathbf{S}_1 = \frac{\partial W}{\partial \mathbf{E}}, \quad \mathbf{S}_2 = \frac{\partial W}{\partial \mathbf{K}}. \quad (22)$$

Remark:

The introduced strain and stress resultant tensors are work-conjugated.

Boundary-Value Problems

Equilibrium equations are

$$\nabla_{\mathcal{X}} \cdot \mathbf{T}_{\mathcal{X}} + \mathbf{f} = \mathbf{0}, \quad (23)$$

$$\nabla_{\mathcal{X}} \cdot \mathbf{M}_{\mathcal{X}} + [\mathbf{F}^T \cdot \mathbf{T}_{\mathcal{X}}]_{\mathcal{X}} + \mathbf{m} = \mathbf{0}. \quad (24)$$

Equations of motion are supplemented by the boundary conditions

$$\begin{aligned} \text{on } \omega_1 : \quad \boldsymbol{\rho} &= \mathbf{r}_0(s), & \text{on } \omega_2 : \quad \mathbf{v} \cdot \mathbf{T}_{\mathcal{X}} &= \mathbf{t}(s), \\ \text{on } \omega_3 : \quad \mathbf{Q} &= \mathbf{h}(s), & \text{on } \omega_4 : \quad \mathbf{v} \cdot \mathbf{M}_{\mathcal{X}} &= \boldsymbol{\mu}(s), \end{aligned} \quad (25)$$

with given functions $\mathbf{r}_0(s)$, $\mathbf{h}(s)$, $\mathbf{t}(s)$, and $\boldsymbol{\mu}(s)$.

Isotropic Shell

Physically linear isotropic shell

$$\begin{aligned} 2W &= \alpha_1 \text{tr}^2 \mathbf{E}_{\parallel} + \alpha_2 \text{tr} \mathbf{E}_{\parallel}^2 + \alpha_3 \text{tr} \left(\mathbf{E}_{\parallel} \cdot \mathbf{E}_{\parallel}^T \right) + \alpha_4 \mathbf{N} \cdot \mathbf{E}^T \cdot \mathbf{E} \cdot \mathbf{N} \\ &+ \beta_1 \text{tr}^2 \mathbf{K}_{\parallel} + \beta_2 \text{tr} \mathbf{K}_{\parallel}^2 + \beta_3 \text{tr} \left(\mathbf{K}_{\parallel} \cdot \mathbf{K}_{\parallel}^T \right) + \beta_4 \mathbf{N} \cdot \mathbf{K}^T \cdot \mathbf{K} \cdot \mathbf{N}, \end{aligned} \quad (26)$$

where $\mathbf{E}_{\parallel} \triangleq \mathbf{E} \cdot \mathbf{A}$, $\mathbf{K}_{\parallel} \triangleq \mathbf{K} \cdot \mathbf{A}$. Here \mathbf{S}_1 and \mathbf{S}_2 have the form

$$\mathbf{S}_1 = \alpha_1 (\text{tr} \mathbf{E}_{\parallel}) \mathbf{A} + \alpha_2 \mathbf{E}_{\parallel}^T + \alpha_3 \mathbf{E}_{\parallel} + \alpha_4 (\mathbf{E} \cdot \mathbf{N}) \otimes \mathbf{N}, \quad (27)$$

$$\mathbf{S}_2 = \beta_1 (\text{tr} \mathbf{K}_{\parallel}) \mathbf{A} + \beta_2 \mathbf{K}_{\parallel}^T + \beta_3 \mathbf{K}_{\parallel} + \beta_4 (\mathbf{K} \cdot \mathbf{N}) \otimes \mathbf{N}. \quad (28)$$

Introducing the fourth-order tensors \mathbf{C}_1 and \mathbf{C}_2 by the formulae

$$\begin{aligned} \mathbf{C}_1 &= \alpha_1 \mathbf{A} \otimes \mathbf{A} + \alpha_2 \mathbf{P}_{\alpha} \otimes \mathbf{A} \otimes \mathbf{P}^{\alpha} \\ &+ \alpha_3 \mathbf{P}_{\alpha} \otimes \mathbf{P}_{\beta} \otimes \mathbf{P}^{\alpha} \otimes \mathbf{P}^{\beta} + \alpha_4 \mathbf{P}_{\alpha} \otimes \mathbf{N} \otimes \mathbf{P}^{\alpha} \otimes \mathbf{N}, \end{aligned}$$

$$\begin{aligned} \mathbf{C}_2 &= \beta_1 \mathbf{A} \otimes \mathbf{A} + \beta_2 \mathbf{P}_{\alpha} \otimes \mathbf{A} \otimes \mathbf{P}^{\alpha} \\ &+ \beta_3 \mathbf{P}_{\alpha} \otimes \mathbf{P}_{\beta} \otimes \mathbf{P}^{\alpha} \otimes \mathbf{P}^{\beta} + \beta_4 \mathbf{P}_{\alpha} \otimes \mathbf{N} \otimes \mathbf{P}^{\alpha} \otimes \mathbf{N}, \end{aligned}$$

we re-write (27) and (28) in a compact form $\mathbf{S}_1 = \mathbf{C}_1 : \mathbf{E}$, $\mathbf{S}_2 = \mathbf{C}_2 : \mathbf{K}$.

Simplest Case of Thin Shells

$$\begin{aligned}\alpha_1 &= C\nu, & \alpha_3 &= C(1 - \nu), & \alpha_4 &= \alpha_s C(1 - \nu), \\ \beta_1 &= D\nu, & \beta_3 &= D(1 - \nu), & \beta_4 &= \alpha_t D(1 - \nu)\end{aligned}$$

$$\Rightarrow \quad C = \frac{Eh}{1 - \nu^2}, \quad D = \frac{Eh^3}{12(1 - \nu^2)}.$$

E, ν - Young's modulus and Poisson's ratio of the bulk material, h - shell thickness. α_s is an analogy to the shear correction factor introduced by Reissner ($\alpha_s = 5/6$) or Mindlin ($\alpha_s = \pi^2/12$). There are other suggestions^{4,5,6}.

⁴J. W. Hutchinson. Shear coefficients for Timoshenko beam theory. *Trans. ASME. J. Appl. Mech.*, 68(1):87-92, 2000

⁵F. Gruttmann and W. Wagner. Shear correction factors in Timoshenko's beam theory for arbitrary shaped cross-sections. *Comput. Mech.*, 27:199-207, 2001

⁶M. B. Rubin. On the quest for the best Timoshenko shear coefficient. *Trans. ASME. J. Appl. Mech.*, 70(1):154-157, 2003

Constraints

- The shell is inextensible in two directions: $\mathbf{D}_1 \cdot \mathbf{E}\mathbf{D}_1 = 0$, $\mathbf{D}_2 \cdot \mathbf{E}\mathbf{D}_2 = 0$.
- It is not shear deformable, $\mathbf{E}\mathbf{n} = \mathbf{0}$, and $\mathbf{D}_1 \cdot \mathbf{E}\mathbf{D}_2 = 0$.
- \mathbf{d}_3 is normal to the base surface, $\mathbf{d}_3 = \mathbf{n}$.

Results: Kirchhoff plate with constraints

- \mathbf{T} should be determined from the equilibrium equations.
- The strain energy has the form $W = W(\mathbf{K})$.
- For plane deformations $W = W(\varphi'_{,1}, \varphi'_{,2})$.

Conclusion

The homogenized model of an elastic network with rigid junctions is a particular case of the nonlinear six-parameter shell model with the energy

$$W = \mathcal{U}(\mathbf{D}_1\mathbf{K}) + \mathcal{U}(\mathbf{D}_2\mathbf{K}).$$

Sheet of beam lattice

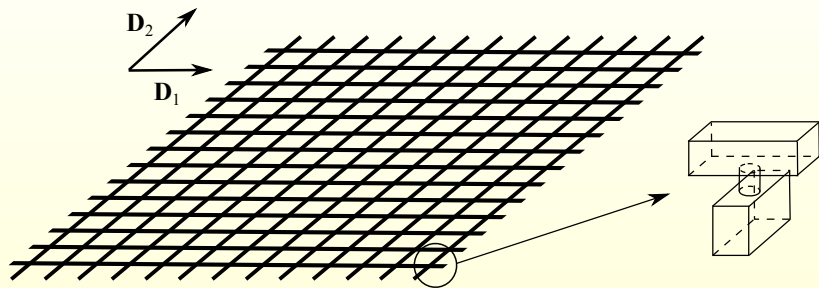


Figure: Scheme of a pantographic sheet and beams connection trough a pivot.

Difference: more degrees of freedom

Shear energy

Presence of elastic pivots, perfect that is with free torsional deformations, or elastic with shear energy bring us more degrees of freedom. So, detailed description of a pantographic lattice requires certain extension of the six-parameter shells model.

Conclusions

- We consider static deformations of a particular class of elastic networks made of orthogonal flexible inextensible fibers with stiff junctions.
- Comparing this structure with the six-parameter (Cosserat, micropolar) nonlinear shell model we concluded that an elastic network can be characterized using this variant of shells theory.
- It gives an example of identification of shell constitutive equations.
- The related shell model can be treated as a thin solid which permits so-called quasi-rigid deformations which are similar to known isometric deformations in the theory of surfaces.

Thank you for your attention!

Further Questions:

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