Factorizations in monoids and rings

Laura Cossu
University of Graz

Many problems in algebra involve the decomposition of certain elements of a ring (or more generally of a monoid) into a product of certain other elements (hereinafter generically referred to as building blocks) that are in some sense minimal. The classical theory of factorization investigates factorizations in which the building blocks are atoms, i.e., non-unit elements of a monoid that are not products of two non-units. For example, it is well known that every non-zero non-unit of a Dedekind domain (more generally, of a Noetherian domain) can be written as a finite product of atoms and that in general such decompositions are not unique. On the other hand, examples of factorizations that lie beyond the scope of the classical theory include additive decompositions into multiplicative units in rings; cyclic decompositions of permutations in the symmetric group of degree n; idempotent factorizations of the "singular elements" of a monoid; and so on.

After an introductory overview of the history and main results in the classical framework, in this course, we combine the language of monoids and preorders (partial orders that are not necessarily antisymmetric) to make first steps towards the construction of a "unified theory of factorization". In particular, we prove an abstract existence theorem that recover, among others, a classical theorem of Cohn on atomic factorizations in cancellative monoids, a classical result by Anderson and Valdes-Leon on "irreducible factorizations" in commutative monoids, but also a theorem by Erdos on idempotent factorizations of square singular matrices over fields. We also introduce a notion of "minimal factorization", suitable for the new abstract setting, and we qualify and quantify the related non-uniqueness properties in specific cases but also in some generality. Examples will help to motivate and illustrate the theory.

Prerequisites: Standard knowledge of (basic) algebraic structures and (binary) relations.

References

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[4] S. Tringali, *An Abstract Factorization Theorem and Some Applications*, J. Algebra 602 (2022), 352–380.